CSC263 Tutorial #1

Runtime Analysis

January 13, 2023

I can't believe it's 2023 already

IMPORTANT ANNOUNCEMENTS

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Figure: when Quer*us forgets to send quiz notification and I miss quiz

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- **DO NOT reopen the tutorial quiz after you submit!!!!!!**

IMPORTANT ANNOUNCEMENTS

- Assignment 1 is also out. This assignment **can be done in pairs**.
- Pre-survey is also out. Get the free 1% of your final grade![1]

[1] but you also have to do the post-survey.

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- How to access tutorials and tutorial quizzes on Quer*us.
- How to set up runtime analysis problems by defining the sample space and probability distribution.
- How to break a complicated random variable into indicator random variables.
- How to perform runtime analysis, given the above.

A bit about myself?

Hi! I'm Paul Zhang, some 5th year student studying math/cs.

- Contact: pol.zhang@utoronto.ca
- Website (slides are posted here!): sjorv.github.io
- Hobbies: Gaming, taking naps at inappropriate times

Not my cat.

- Favourite food: sushi juice

Question time!

(I would probably recommend you view this on Quer*us if you have a computer).

Consider the following algorithm to find the maximum element in a list.

```
1 FIND-MAX(L):
2     max = -oo # minus infinity
3     for k = 0 to len(L)-1:
4         if L[k] > max:
5         max = L[k]
6     return max
```

For this problem, we are interested in the number of times that variable max gets assigned a value (Lines 2 and 5!).

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Why? The number of times `max` gets assigned a value only depends on the relative rank of each element of `L`, rather than the values of each element. For example, for both inputs `L = [3, 2, 1, 4]` and `L = [300, 190, -60, 9999]`, `max` is assigned 3 times.

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Sample space for inputs of size n:

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Question: For this problem, what is a reasonable probability distribution over S_n?
Answer: In this problem, there is no reason to assign more probability to some inputs over others. So we choose the uniform distribution over S_n.

Best/Worst case

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Question: Out of all inputs in S_n, what is the best case number of times that `max` is assigned a value? For which input(s) does this best case occur?
Question: Out of all inputs in S_n, what is the worst case number of times that `max` is assigned a value? For which input(s) does this worst case occur?

Average case

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Question: Assume a uniform distribution of inputs over S_n. What is the average number of times that `max` is assigned a value?

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Question: What are the possible values of X (in terms of n)?
Ans: The possible values are $2, 3, \ldots, n, n + 1$. We must assign to `max` at least 2 times, and at most $n + 1$ times (see best and worst-case analysis).

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$$\mathbb{E}(X) = \sum_{k=2}^{n+1} kP(X = k).$$

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Question: What is $P(X = k)$?

This is a bit difficult to calculate...

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For $i = 1, 2, \ldots, n$, Define

$$X_i = \begin{cases} 1 & \text{On the } i\text{th iteration of the loop, } \texttt{max} \text{ is reassigned.} \\ 0 & \text{On the } i\text{th iteration of the loop, } \texttt{max} \text{ isn't reassigned.} \end{cases}$$

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Task: Let $n = 5$, and `L = [1, 3, 2, 5, 4]`. Find X_1, X_2, X_3, X_4, X_5.

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Answer: $X_1 = 1$, $X_2 = 1$, $X_3 = 0$, $X_4 = 1$, $X_5 = 0$.

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Answer: $X_1 = 1$, $X_2 = 1$, $X_3 = 0$, $X_4 = 1$, $X_5 = 0$.

Notice this!

$$X = 1 + X_1 + X_2 + \ldots + X_n.$$

(The 1 at the beginning counts the assignment `max = -oo`.)

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Task: Find $P(X_i = 1)$ in terms of i. Then compute $E(X)$.

Quiz time!!!

Now do the quiz on Quer*us!