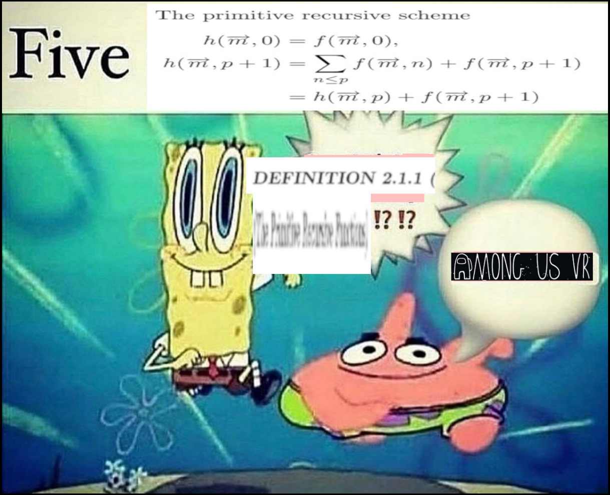
CSC363 Tutorial #2

Primitive Recursive Functions



January 25, 2023

Things covered in this tutorial

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- What's a "function"?
- What are the primitive recursive functions?
- How can I use composition and primitive recursion?
- Could I get a hint for A1 Q3? No, but you may cite these slides for your homework. (You still have to provide a formal proof that the functions in this tutorial are PRIM.)[1]

[1] Citation: Paul "sjorv" Zhang. "CSC363 Tutorial #2. Primitive Recursive Functions. Five the primitive recursive scheme among us vr.". Chungus Publishing, 2023.

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($\mathbb{N}$ includes 0 in this course.)

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   - The successor function $S : \mathbb{N} \to \mathbb{N}$, $S(n) = n + 1$.
   - For each $k \in \mathbb{N}$ and $i = 1, \ldots, k$, the projection function $C_i^k : \mathbb{N}^k \to \mathbb{N}$, $C_i^k(n_1, n_2, \ldots, n_k) = n_i$.[2]

[2] In other words, C_i^k takes in k arguments, and returns the ith one.

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   - Composition: If $g : \mathbb{N}^k \to \mathbb{N}$ and $h_1, h_2, \ldots, h_k : \mathbb{N}^\ell \to \mathbb{N}$ are in PRIM, then $f : \mathbb{N}^\ell \to \mathbb{N}$ given by[3]

     $$f(\vec{m}) = g(h_1(\vec{m}), \ldots, h_k(\vec{m}))$$

     is primitive recursive.

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   - Primitive Recursion: If $g : \mathbb{N}^\ell \to \mathbb{N}$ and $h : \mathbb{N}^{\ell+2} \to \mathbb{N}$ are in PRIM, then $f : \mathbb{N}^{\ell+1} \to \mathbb{N}$ given by

     $$f(\vec{m}, 0) = g(\vec{m})$$
     $$f(\vec{m}, n+1) = h(\vec{m}, n, f(\vec{m}, n))$$

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is primitive recursive.
In other words, you can use for-loops.

```
f(m, n):
  curr = g(m)
  for i in 1..n:
    curr = h(m, i-1, curr)
  return curr
```

This is very powerful!

Addition is in PRIM

The function $+ : \mathbb{N}^2 \to \mathbb{N}$, $f(m,n) = m + n$ is in PRIM, due to primitive recursion:

```
+(m, n):
  curr = C^1_1(m)
  for i in 1..n:
    curr = S(C^3_3(m, i-1, curr))
  return curr
```

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Explanation:

- C_1^1 is the identity function: $C_1^1(m) = m$. [4]
- `curr = S(C^3_3(m, i-1, curr))` adds 1 to `curr`:

$$C_3^3(m, i-1, \texttt{curr}) = \texttt{curr}$$

$$S(\texttt{curr}) = \texttt{curr} + 1$$

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Formally,

$$\begin{aligned} +(m, 0) &= C_1^1(\vec{m}) \\ +(m, n+1) &= S(C_3^3(m, n, +(m, n))). \end{aligned}$$

Multiplication is in PRIM

Your turn! Show that the function $f : \mathbb{N}^2 \to \mathbb{N}$, $f(m, n) = m \cdot n$ is in PRIM.

Multiplication is in PRIM

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Here's the template for primitive recursion again:

```
f(m, n):
  curr = g(m)
  for i in 1..n:
    curr = h(m, i-1, curr)
  return curr
```

Hint: You know that the addition function $+ : \mathbb{N}^2 \to \mathbb{N}$, $+(m, n) = m + n$ is in PRIM now.

If you're finished, try showing $f : \mathbb{N}^2 \to \mathbb{N}$, $f(m, n) = m^n$ is in PRIM too.

More PRIM functions!

Show that $f : \mathbb{N}^2 \to \mathbb{N}$, $f(m, n) = m - n$ is in PRIM.

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Try this instead. **Task:** Show that the "try to subtract 1" function

$$\delta : \mathbb{N} \to \mathbb{N}, \delta(m) = \begin{cases} m - 1 & m > 0 \\ 0 & m = 0 \end{cases}$$

is in PRIM.

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Hint 1: Try to prove $f : \mathbb{N}^2 \to \mathbb{N}$, $f(m, n) = \delta(n)$ is in PRIM.

```
f(m, n):
  curr = g(m)
  for i in 1..n:
    curr = h(m, i-1, curr)
  return curr
```

Hint 2: In the above, h won't need the value of m or `curr`.

More PRIM functions!

Task: Show that the “try to subtract” function

$$\dot{-} : \mathbb{N}^2 \to \mathbb{N}, \dot{-}(m, n) = \begin{cases} m - n & m \geq n \\ 0 & m < n \end{cases}$$

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Hint: Use the "try to subtract 1" function δ.

More PRIM functions!

Task: Show that the "is greater than zero?" function

$$\mathrm{sg} : \mathbb{N} \to \mathbb{N}, \mathrm{sg}(n) = \begin{cases} 0 & n = 0 \\ 1 & n > 0 \end{cases}$$

is in PRIM.

```
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  for i in 1..n:
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```

More PRIM functions!

Task: Show that the "is equal than zero?" function

$$\overline{\mathrm{sg}} : \mathbb{N} \to \mathbb{N}, \overline{\mathrm{sg}}(n) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

is in PRIM.
Hint: You could do this using primitive recursion, but a shorter solution would be to use sg.

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Note that sg, $\overline{\mathrm{sg}}$ allow you to use "if-statements".

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m \ n	0	1	2	3	4	n
0	1	2	3	4	5	$n+1$
1	2	3	4	5	6	$n+2 = 2+(n+3)-3$
2	3	5	7	9	11	$2n+3 = 2\cdot(n+3)-3$
3	5	13	29	61	125	$2^{(n+3)}-3$
4	13 $= 2^{2^2}-3$ $= 2\uparrow\uparrow 3-3$	65533 $= 2^{2^{2^2}}-3$ $= 2\uparrow\uparrow 4-3$	$2^{65536}-3$ $= 2^{2^{2^{2^2}}}-3$ $= 2\uparrow\uparrow 5-3$	$2^{2^{65536}}-3$ $= 2^{2^{2^{2^{2^2}}}}-3$ $= 2\uparrow\uparrow 6-3$	$2^{2^{2^{65536}}}-3$ $= 2^{2^{2^{2^{2^{2^2}}}}}-3$ $= 2\uparrow\uparrow 7-3$	$\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n+3}-3$ $= 2\uparrow\uparrow(n+3)-3$
5	65533 $= 2\uparrow\uparrow(2\uparrow\uparrow 2)-3$ $= 2\uparrow\uparrow\uparrow 3-3$	$2\uparrow\uparrow\uparrow 4-3$	$2\uparrow\uparrow\uparrow 5-3$	$2\uparrow\uparrow\uparrow 6-3$	$2\uparrow\uparrow\uparrow 7-3$	$2\uparrow\uparrow\uparrow(n+3)-3$
6	$2\uparrow\uparrow\uparrow\uparrow 3-3$	$2\uparrow\uparrow\uparrow\uparrow 4-3$	$2\uparrow\uparrow\uparrow\uparrow 5-3$	$2\uparrow\uparrow\uparrow\uparrow 6-3$	$2\uparrow\uparrow\uparrow\uparrow 7-3$	$2\uparrow\uparrow\uparrow\uparrow(n+3)-3$
m	$(2\to 3\to(m-2))-3$	$(2\to 4\to(m-2))-3$	$(2\to 5\to(m-2))-3$	$(2\to 6\to(m-2))-3$	$(2\to 7\to(m-2))-3$	$(2\to(n+3)\to(m-2))-3$

It grows too quickly to be captured using for-loops.

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Proof (hard!): show that the set of functions

$$\mathcal{A} = \{f : \exists t \in \mathbb{N}, \forall x_1, \ldots, x_n \in \mathbb{N}, f(x_1, \ldots, x_n) < A(t, \max_i x_i)\}$$

contains all PRIM functions, via structural induction.

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contains all PRIM functions, via structural induction. This shows that $A(m, n)$ grows strictly faster than any PRIM function, and hence cannot be PRIM itself.

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To avoid spoiling content, I am legally required to not speak any further on the halting problem.

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