CSC363 Tutorial #3

Decidable and Recognizable sets

February 1, 2023

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- ⋆ How do I show that something is decidable or recognizable?

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- ⋆ What's an enumerator?
- ⋆ Can I get a hint for A2?

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- ⋆ How do I show that something is decidable or recognizable?
- ⋆ What's an enumerator?
- ⋆ Can I get a hint for A2? No, but you may cite these slides for your homework. (You still have to prove everything.)[1]

[1] Citation: Paul "sjorv" Zhang. "Sussy Tutorial #3. Decidable and Recognizable Sets.". EXTRACT_MONEY_FROM_STUDENTS Publishing, 2023.

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For this tutorial, we will use the Church-Turing Thesis to write pseudocode instead of low-level TMs.

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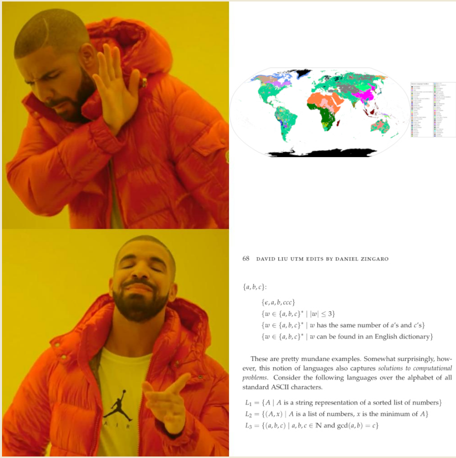
Let Σ be an **alphabet**. Then Σ^* is the set of all finite strings using characters from Σ. A **language** is any subset of Σ^*.



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We say L is recognizable if there is a Turing machine M (called the recognizer) such that for any input $s \in \Sigma^*$:

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Note the difference! M always has to halt in order to be a decider.

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Example: Let $\Sigma = \{0, 1\}$, and $L = \{0^n 1^n : n \in \mathbb{N}\}$.[2] Show that L is decidable.

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Proof. The following is the pseudocode of a decider M for L:

```
M(w):
  n = length(w)
  if n is odd:
    reject
  for i in 0 to (n/2 - 1):
    if w[n/2] != 0 or w[n/2 + i] != 1:
      reject
  accept
```

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Decidable and Recognizable

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Here are some synonyms for recognizable:

- ⋆ Listable.
- ⋆ Recursively enumerable (r.e.).
- ⋆ Computably enumerable (c.e.).
- ⋆ Partially decidable.
- ⋆ Σ^0_1.

Worksheet time!

Try doing Exercise -1. Here's the definition of *decidable* again:

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Try doing Exercise -1. Here's the definition of *decidable* again:

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If you are done, try Exercise 0 as well.

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Are there any non-decidable languages?

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Decidable Languages

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[4] Recall that a Diophantine equation is a multivariable equation like $x^3 + 3xyz - w^{420} = 2$.

Decidable Languages

Are there any non-decidable languages? Yes.

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Source: trust me bro.[5]

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[5] You can look up "Hilbert's tenth problem".

Recognizable Languages

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Source:

```
M(p):
  if p isn't a valid Diophantine equation:
    reject
  n = number of variables in p
  s = 0
  while True:
    for all x1, x2, ..., xn
    with x1 + x2 + ... + xn = s:
      if (x1, x2, ... xn) is a solution to p:
        accept
      s += 1
```

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For example, if p is the equation $3x^5 - xy + y^2 = 3$, $M(p)$ will:

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For example, if p is the equation $3x^5 - xy + y^2 = 3$, $M(p)$ will:

- ⋆ Check if $x = 0, y = 0$ is a solution. If not,
- ⋆ Check if $x = 0, y = 1$ is a solution. If not,
- ⋆ Check if $x = 1, y = 0$ is a solution. If not,
- ⋆ Check if $x = 0, y = 2$ is a solution. If not,
- ⋆ Check if $x = 1, y = 1$ is a solution. If not,
- ⋆ Check if $x = 2, y = 0$ is a solution. If not,
- ⋆ Check if $x = 0, y = 3$ is a solution. If not,
- ⋆ Check if $x = 1, y = 2$ is a solution. If not,
- ⋆ Check if $x = 2, y = 1$ is a solution. If not,
- ⋆ Check if $x = 3, y = 0$ is a solution. If not,
- ⋆ Check if $x = 0, y = 4$ is a solution. If not,
- ⋆ Check if $x = 1, y = 3$ is a solution. If not,
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- ⋆ Check if $x = 4, y = 0$ is a solution. If not,
- ⋆ Check if $x = 0, y = 5$ is a solution. If not,
- ⋆ Check if $x = 1, y = 4$ is a solution. If not,

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If $3x^5 - xy + y^2 = 3$ has no natural solutions, $M(p)$ will

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If $3x^5 - xy + y^2 = 3$ has a natural solution $(x, y) \in \mathbb{N}^2$, $M(p)$ will eventually accept.

If $3x^5 - xy + y^2 = 3$ has no natural solutions, $M(p)$ will run forever, i.e. loop.

Recognizable Languages

Question: Why couldn't have $M(p)$ done the following sequence of checks, with p being the equation $3x^5 - ky + y^2 = 3$?

- ⋆ Check if $x = 0, y = 0$ is a solution. If not,
- ⋆ Check if $x = 0, y = 1$ is a solution. If not,
- ⋆ Check if $x = 0, y = 2$ is a solution. If not,
- ⋆ Check if $x = 0, y = 3$ is a solution. If not,
- ⋆ Check if $x = 0, y = 4$ is a solution. If not,
- ⋆ Check if $x = 0, y = 5$ is a solution. If not,
- ⋆ Check if $x = 0, y = 6$ is a solution. If not,
- ⋆ Check if $x = 0, y = 7$ is a solution. If not,
- ⋆ Check if $x = 0, y = 8$ is a solution. If not,
- ⋆ Check if $x = 0, y = 9$ is a solution. If not,
- ⋆ Check if $x = 0, y = 10$ is a solution. If not,
- ⋆ Check if $x = 0, y = 11$ is a solution. If not,
- ⋆ Check if $x = 0, y = 12$ is a solution. If not,
- ⋆ Check if $x = 0, y = 13$ is a solution. If not,
- ⋆ Check if $x = 0, y = 14$ is a solution. If not,
- ⋆ Check if $x = 0, y = 15$ is a solution. If not,

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Some non-recognizable languages:

- ⋆ $L = \{M\#x : M \text{ is a TM that loops on } x\}$.
- ⋆ $L = \{M : M \text{ is a TM that halts on all inputs}\}$.

Enumerators (possibly useful for assignment!)

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Example: the following program is an enumerator for the language $\{0^n 1^n : n \in \mathbb{N}\}$.

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def enum():
  n = 0
  while True:
    print('0'*n + '1'*n)
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Try running `enum.py` on your computer!

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Theorem (Sipser 3.21)

A language L is recognizable if and only if it has an enumerator.

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Proof. First, suppose L has an enumerator `enum`. Define M as follows:

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M(x):
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- ⋆ If $x \notin L$, then x is never printed, and $M(x)$ loops.

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- ⋆ If $x \in L$, then x is eventually printed by the enumerator, and $M(x)$ accepts.
- ⋆ If $x \notin L$, then x is never printed, and $M(x)$ loops.

Thus M recognizes L.

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Proof. Now, suppose L has a recognizer M. Define `enum` as follows:

```
enum():
  n = 0
  while True:
    for all strings w of length <= n:
        run M(w)
        if M(w) accepts:
            print(w)
    n += 1
```

Question: The above is not a valid enumerator of L. Why?

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Task:

- ⋆ Given any $w \notin L$, Why does `enum()` never print out w?
- ⋆ Given any $w \in L$, Why does `enum()` eventually print out w?

Useful trick

```
enum():
  n = 0
  while True:
    for all strings w of length <= n:
        run M(w) for n steps
        if M(w) accepts:
            print(w)
    n += 1
```

This idea of "running for *n* steps, then increasing the maximum time allowed and trying again" is very useful in CSC363!