Things to remember

- Let Σ be the input alphabet. We let Σ^* denote the set of finite strings containing characters from Σ. For example, suppose $\Sigma = \{\mathrm{a}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{i}, \mathrm{m}\}$. Then Σ^* contains the strings $\mathrm{mmmmmddddddd}$, mid, $\mathrm{academia}$, $\mathrm{macademia}$, and the empty string ϵ, but does not contain the strings bad or $13\mathrm{d}$.
- A subset of Σ^* is called a **language**. With $\Sigma = \{\mathrm{a}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{i}, \mathrm{m}\}$ again, examples of languages include the empty language $\emptyset$, the universal language Σ^*, the singleton language $\{\mathrm{a}\}$, and so on.
- A language L is **decidable** if there is a Turing machine M such that for any input $s \in \Sigma^*$:
  - If $s \in L$, then M accepts s.
  - If $s \notin L$, then M rejects s.
- A language L is **recognizable** if there is a Turing machine M such that for any input $s \in \Sigma^*$:
  - If $s \in L$, then M accepts s.
  - If $s \notin L$, then M rejects or loops on s.
- A function $f : \Sigma^* \to \Sigma^*$ is **computable** if there is a Turing machine M such that for any input $s \in \Sigma^*$:
  - M halts (either accepts or rejects).
  - M outputs $f(s)$.

Dealing with an identity crisis

Recognizable/decidable languages are central notions to computability theory. They also go by many names!

- Recognizable languages are sometimes called *recursively enumerable languages*.
- Decidable languages are sometimes called *recursive* or *computable* languages.

Exercises

Exercise -1. Let $\Sigma = \{0, 1\}$ be the input alphabet. Show that the following languages are decidable. Do not formally construct the Turing machines; simply describe how a Turing machine (or equivalently, a computer program) would decide if an input is in the language.

- $\emptyset$.
- $\{ww : w \in \Sigma^*\}$.
- $\{w \in \Sigma^* : w \text{ is a prime number in binary}\}$.
- $\{w \in \Sigma^* : \text{There is an integer solution } x \text{ to } wx^6 - 5wx^5 + 4000x^4 - 2wx - 60 = 0\}$.

Exercise 0. Show that a language L is decidable if and only if its *characteristic function* χ_L is computable, where

$$\chi_L(s) = \begin{cases} 1, & s \in L, \\ 0, & s \notin L. \end{cases}$$

Exercise 1. Let L_1, L_2 be languages.

(a) If L_1 and L_2 are decidable, prove that the following sets are decidable, by giving a high-level description/pseudocode of an appropriate Turing machine.

(i) $L_1 \cup L_2$. (ii) $L_1 \cap L_2$. (iii) L_1^c.

(b) If L_1 and L_2 are recognizable, prove that the following sets are recognizable.

(i) $L_1 \cup L_2$. (ii) $L_1 \cap L_2$. (iii) $L_1L_2 = \{st : s \in L_1,\ t \in L_2\}$.

(c) Is the complement of a recognizable language always recognizable? If not, when is this true?

Exercise 2. Show that every infinite recognizable language has an infinite decidable subset.