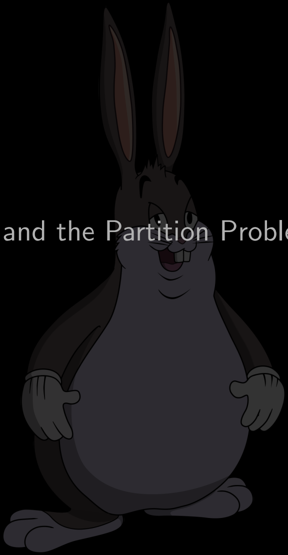
CSC363 Tutorial #10

Subset Sum, Partition problem

March 30, 2022

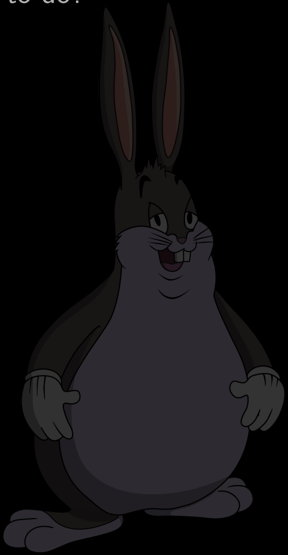
Learning objectives this tutorial



- Review the Subset Sum Problem.
- Introduce the Partition problem.
- Prove that the Subset Sum Problem and the Partition Problem p-reduce to each other.

Subset Sum Review

Task: Recall the (integer, multiset) Subset Sum (Decision) Problem. What are you given? What are you asked to do?



[1] A multiset is a set that allows duplicates.

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Ans: We are given a finite (multi)set[1] of integers $S = \{x_1, x_2, \ldots, x_n\}$ and a "target" $t \in \mathbb{Z}$. We are asked to determine whether there is a subset $S' \subseteq S$ such that

$$\sum_{s' \in S'} s' = t.$$

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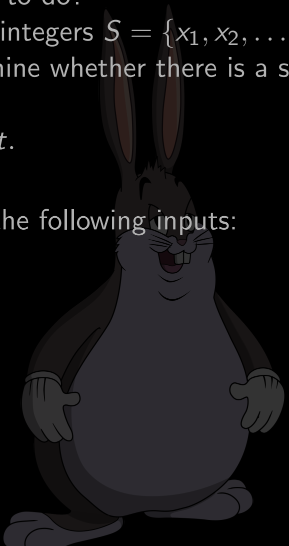
Task: Solve the subset sum problem for the following inputs:

- $S = \{18, 37, 20, 13, 33\}$, $t = 75$.
- $S = \{20, 21, 22, 36, 67\}$, $t = 90$.



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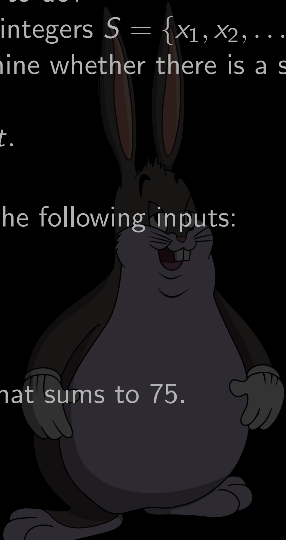
Ans:

- There is a subset $S' = \{18, 37, 20\}$ that sums to 75.
- There is no subset that sums to 90.



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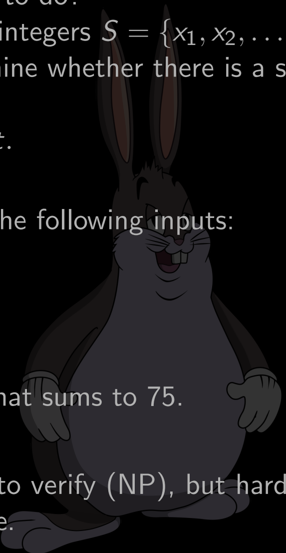
Remind yourself that Subset Sum is easy to verify (NP), but hard to solve (NP-hard), so Subset Sum is NP-complete.



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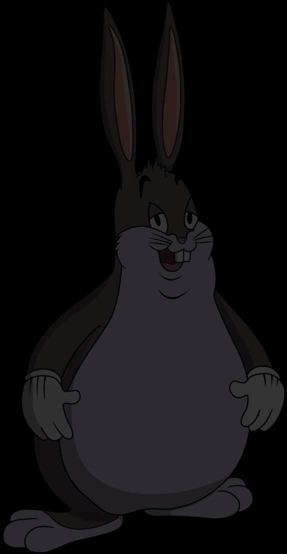
Consider the following scenario:



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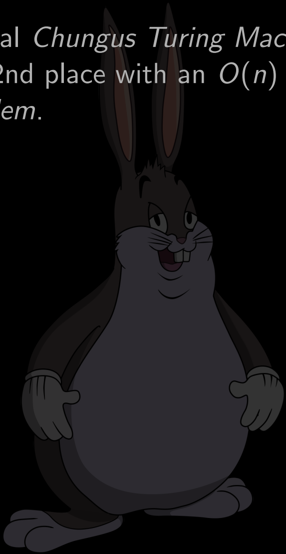
You team up with your friend in the annual *Chungus Turing Machine Contest*. Your team manages to achieve 2nd place with an $O(n)$ solution of the NP-complete *Chungus-Cover Problem*.



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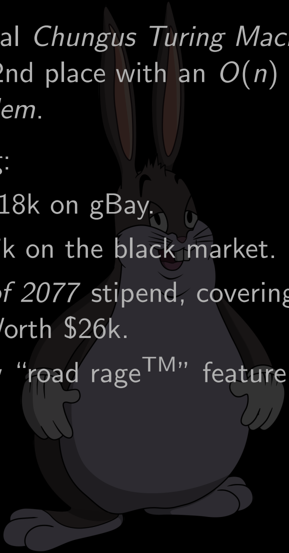

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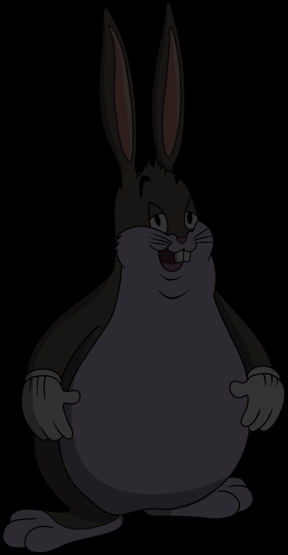

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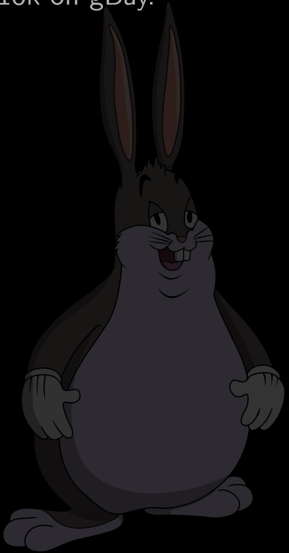

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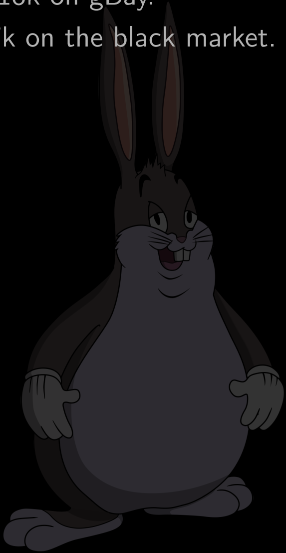

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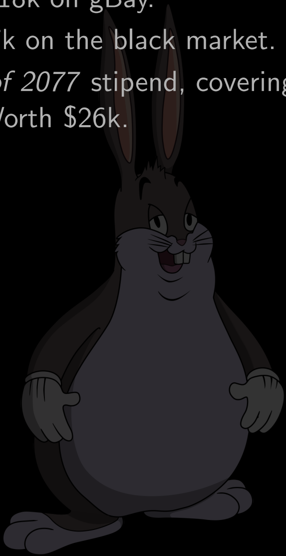

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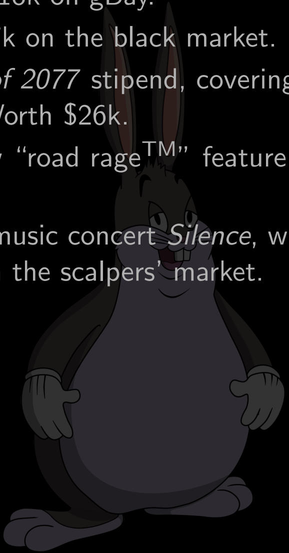

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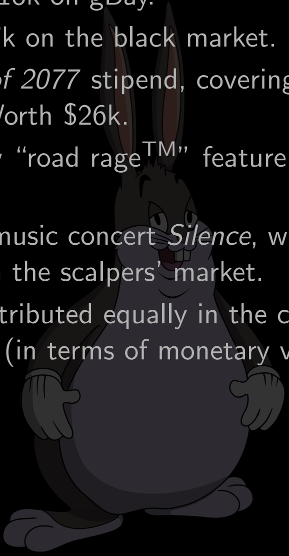

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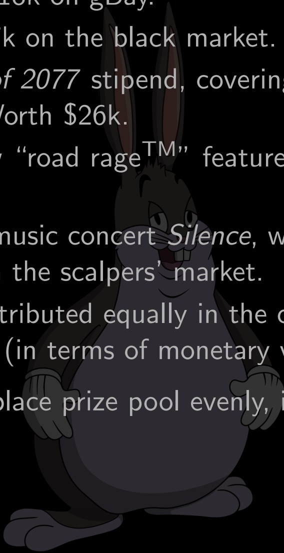

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Ans: Yes. You take the Chungus plushie (\$37k) and the stipend (\$26k), and your friend takes the rest. Your prize is $37\text{k} + 26\text{k} = 53\text{k}$, while your friend's prize is $18\text{k} + 15\text{k} + 15\text{k} + 15\text{k} = 53\text{k}$.

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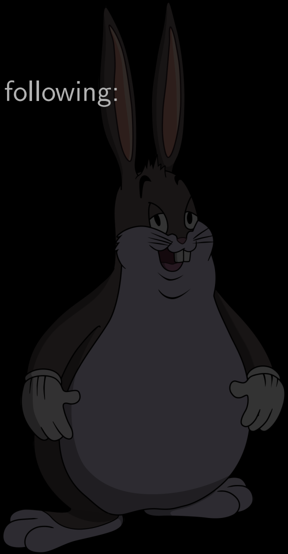
In general, the partition problem asks the following:



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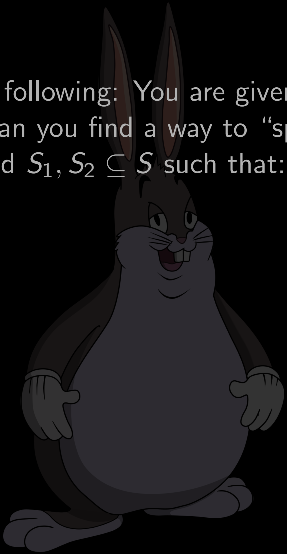

In general, the partition problem asks the following: You are given a finite (multi)set of integers $S = \{s_1, \ldots, s_n\}$. Can you find a way to "split" S into two equal parts? That is, can you find $S_1, S_2 \subseteq S$ such that:

1. $S_1 \cup S_2 = S$,
2. $S_1 \cap S_2 = \emptyset$,
3. $\sum S_1 = \sum S_2$.

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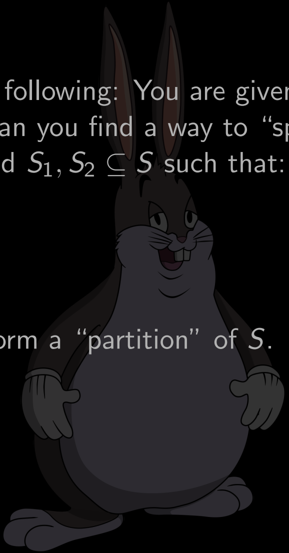
Note that 1. and 2. say that S_1 and S_2 form a "partition" of S.



Partition Problem

Task: Determine if the following sets are partitionable.

1. $S = \{18, 37, 26, 15, 15, 15\}$.
2. $S = \{18, 37, 20, 13, 33\}$.
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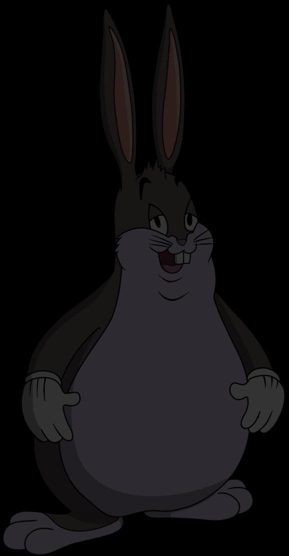

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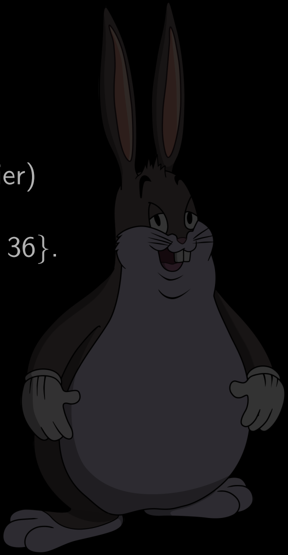

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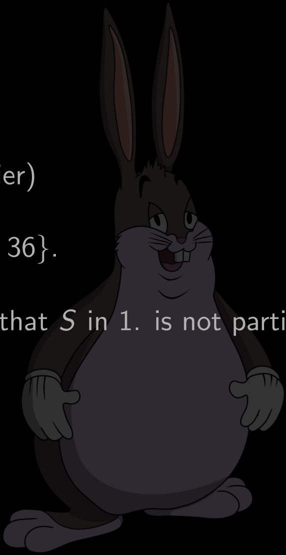

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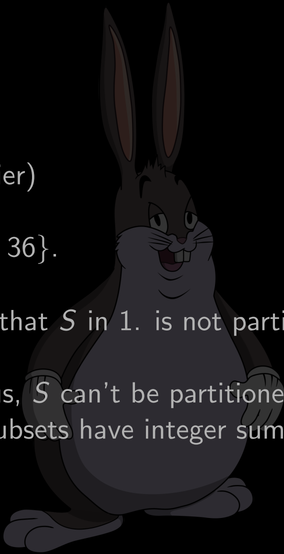
Ans: Notice that $\sum S = 121$ is odd. Thus, S can't be partitioned into two subsets with equal sum (since both subsets have integer sums).



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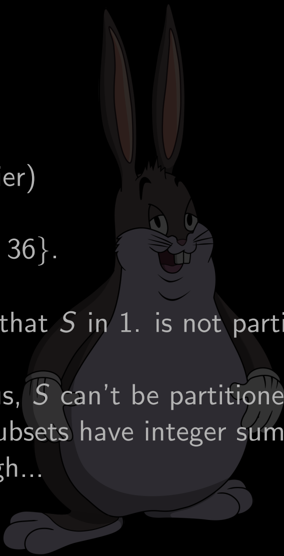

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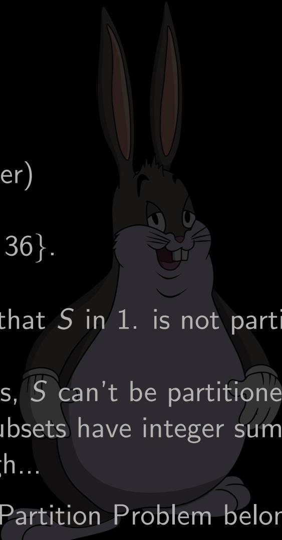

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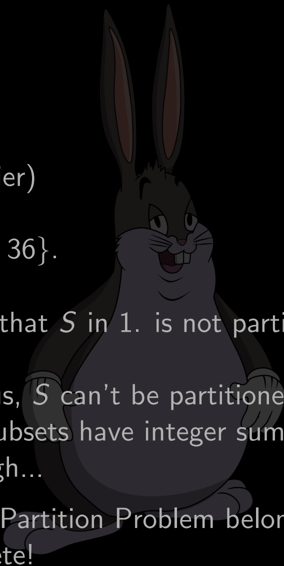

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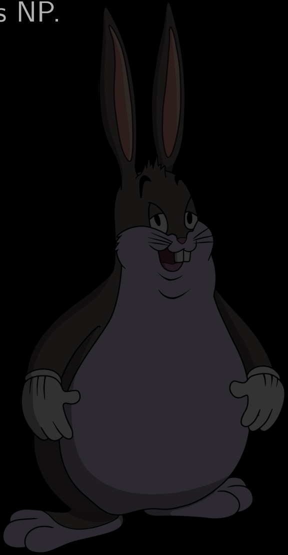
Ans: The partition problem is NP-complete!

Partition Problem is NPC

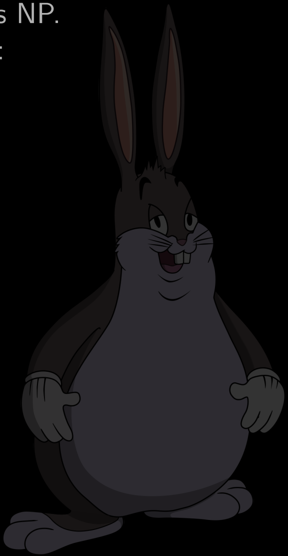
Task: Prove that the Partition Problem is NP.



Partition Problem is NPC

Task: Prove that the Partition Problem is NP.
Ans: We can build a poly-time verifier V:

```
V(S, S1):
  if S1 isn't a subset of S2:
    reject
  S2 = S \ S1
  if sum(S1) = sum(S2):
    accept
  reject
```



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V(S, S1):
  if S1 isn't a subset of S2:
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  S2 = S \ S1
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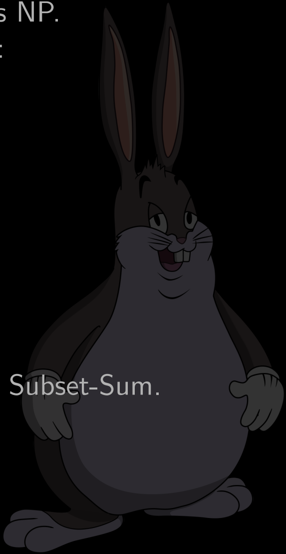
Unrelated Task: Show that Partition $\leq_p$ Subset-Sum.



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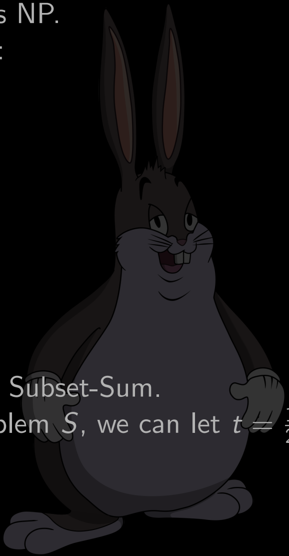
Ans: For an instance of the partition problem S, we can let $t = \frac{1}{2} \sum S$. S is partitionable iff (S, t) is in subset sum.



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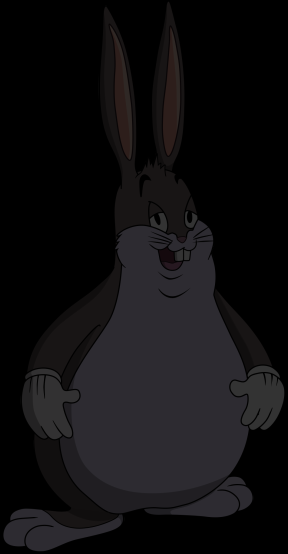
Now we show the Partition Problem is NP-hard, by showing Subset-Sum $\leq_p$ Partition. Let (S, t) be an instance of the Subset-Sum problem.



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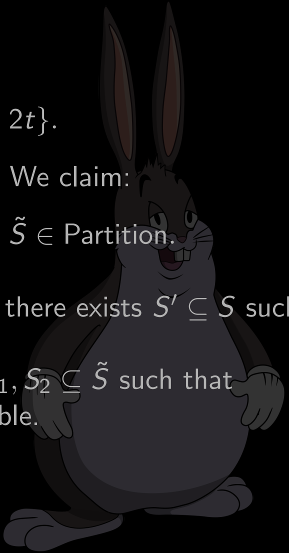
($\Rightarrow$) Suppose $(S, t) \in$ Subset-Sum. Then there exists $S' \subseteq S$ such that $\sum S' = t$.

Task: Using the above, find a partition $S_1, S_2 \subseteq \tilde{S}$ such that $\sum S_1 = \sum S_2$. This shows $\tilde{S}$ is partitionable.



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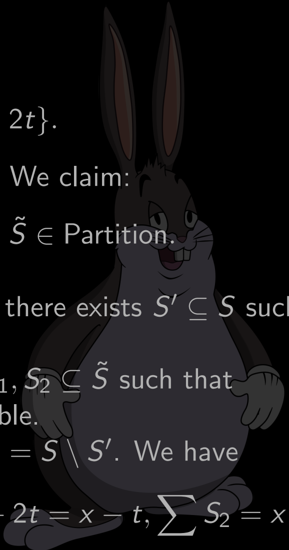

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Ans: Defining $S_1 = S' \cup \{x - 2t\}$ and $S_2 = S \setminus S'$. We have

$$\sum S_1 = (\sum S') + x - 2t = t + x - 2t = x - t, \sum S_2 = x - t.$$

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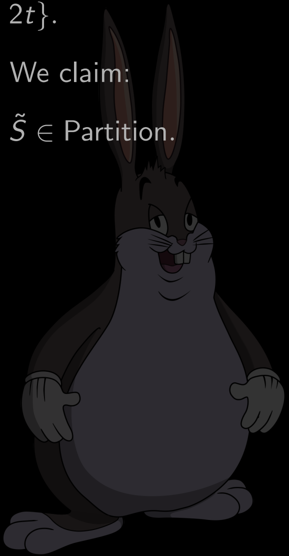

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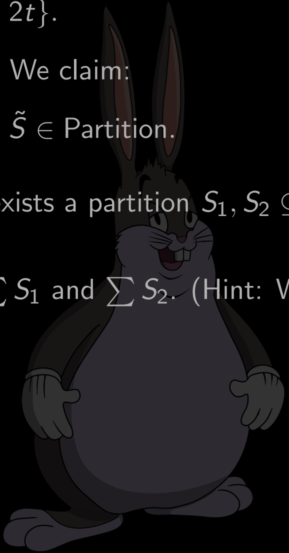

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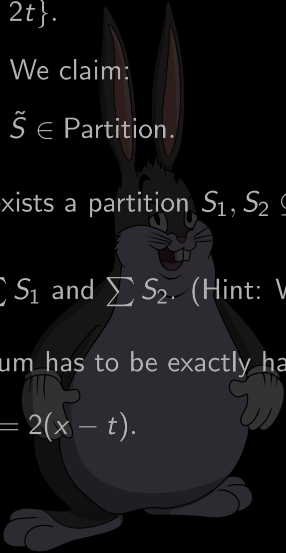
Ans: $\sum S_1 = \sum S_2 = x - t$, since each sum has to be exactly half of

$$\sum \tilde{S} = x + x - 2t = 2(x - t).$$



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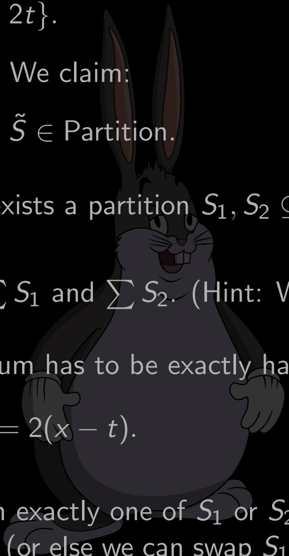

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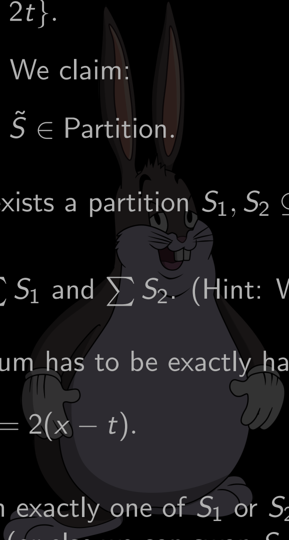

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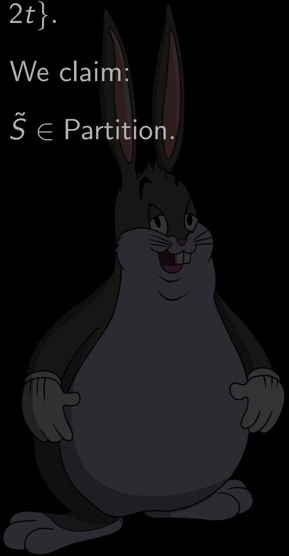

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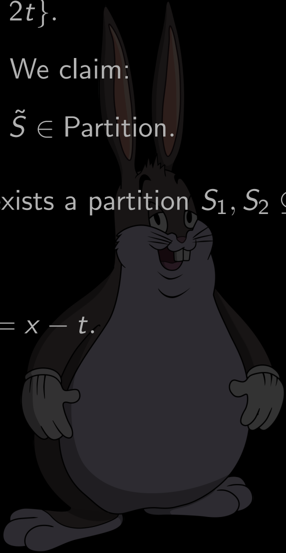

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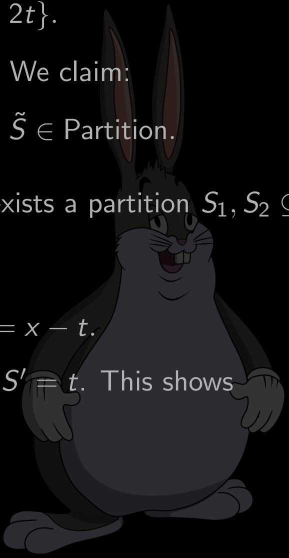

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Ans: Let $S' = S \setminus S_2$. Since $x - 2t \in S_1$, we have $S_2 \subseteq S$, so

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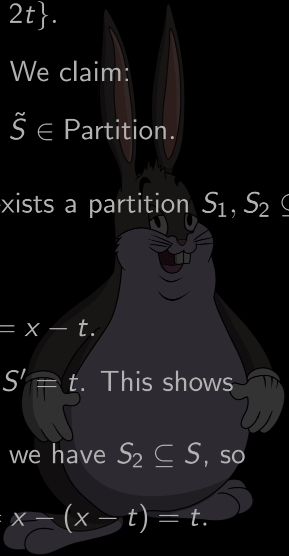

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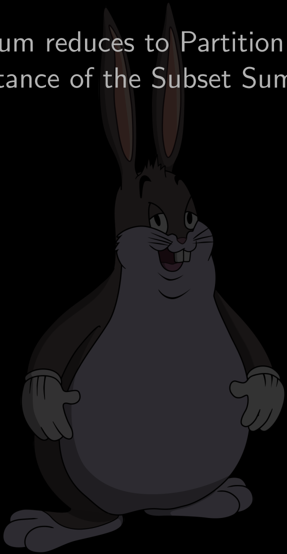

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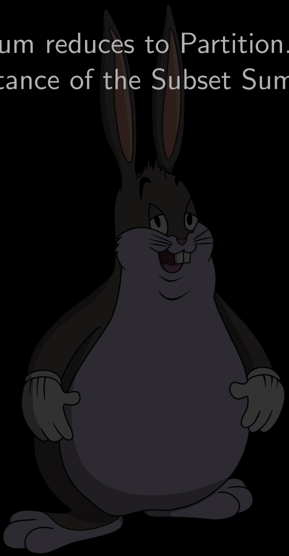

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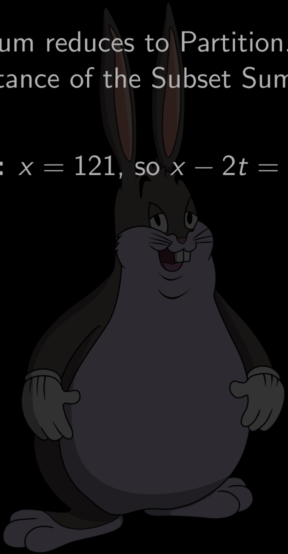

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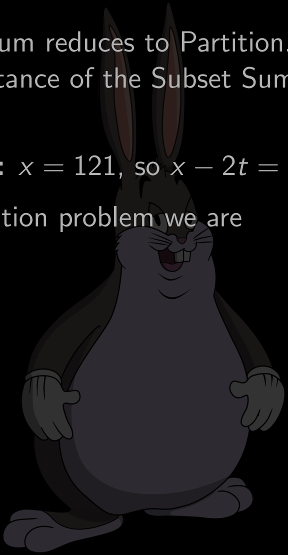

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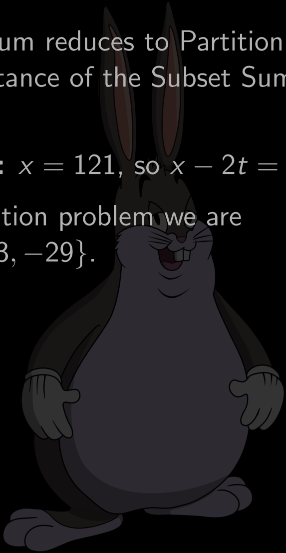

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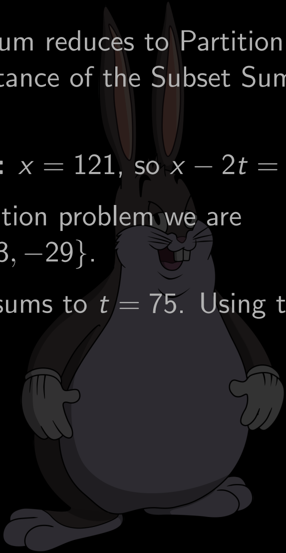

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Ans: Let $S_1 = S' \cup \{x - 2t\} = \{18, 37, 20, -29\}$, and $S_2 = \tilde{S} \setminus S_1 = \{20, 13, 33\}$. Both sum to 46.