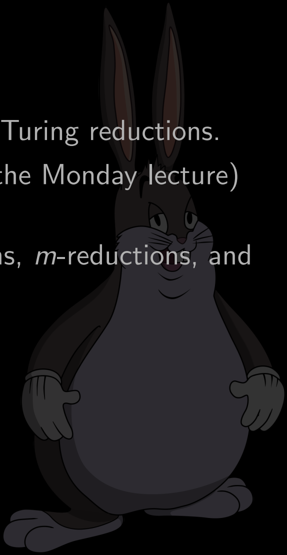
CSC363 Tutorial #4

Turing reductions! (and some assignment feedback)

February 09, 2022

Learning objectives this tutorial

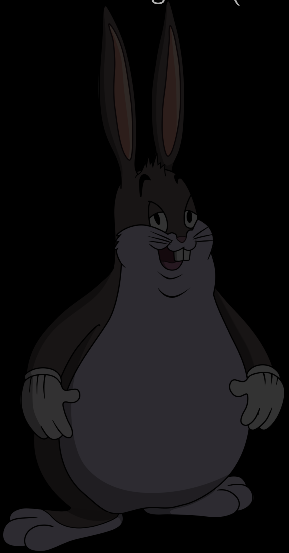
- Review (hopefully, if you remember) Turing reductions.
- Learn (or review, if you've attended the Monday lecture) m-reductions and 1-reductions.
- Distinguish between Turing reductions, m-reductions, and 1-reductions.



Assignment 1 stuff

Assignment 1 feedback has been posted to Pizza. 🍕

Please read through it! Some common mistakes throughout (also appearing on Assignment 2):



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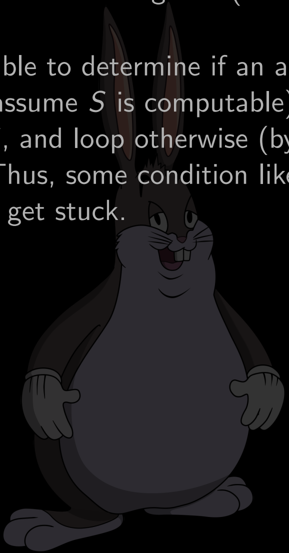
- Given a CE set S, we might not be able to determine if an arbitrary $x \in \mathbb{N}$ is in S or not (unless we can assume S is computable). The best we can do is confirm that $x \in S$, and loop otherwise (by printing out elements of S until we find x). Thus, some condition like "if $x \in S$" might cause your program to get stuck.



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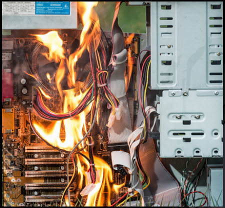
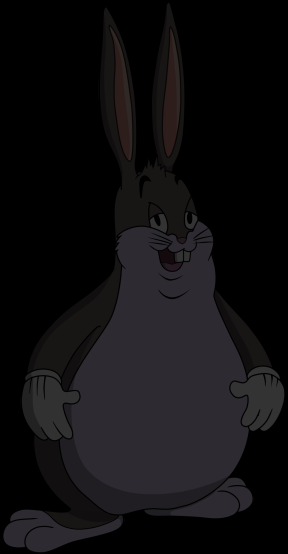
- Given a CE set S, we might not be able to determine if an arbitrary $x \in \mathbb{N}$ is in S or not (unless we can assume S is computable). The best we can do is confirm that $x \in S$, and loop otherwise (by printing out elements of S until we find x). Thus, some condition like "if $x \in S$" might cause your program to get stuck.



- Sometimes, the solution proves a completely different (and maybe more trivial) statement. Make sure to reiterate what you are trying to prove, so that you don't lose track!

Turing reductions!

Task: Show that $K = \{x : \varphi_x(x) \text{ halts}\}$ is computable!



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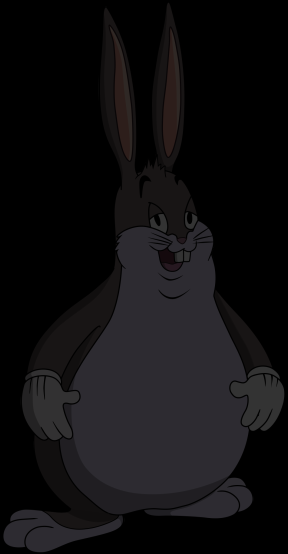
Ans: That's kinda impossible...



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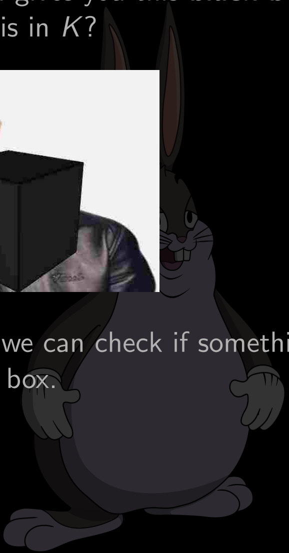

Is K now computable? Yes, because now we can check if something is in K or not by just feeding it into this black box.

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What about $\bar{K}$? Without Eminem's help, $\bar{K}$ is not even CE. Is it now computable?

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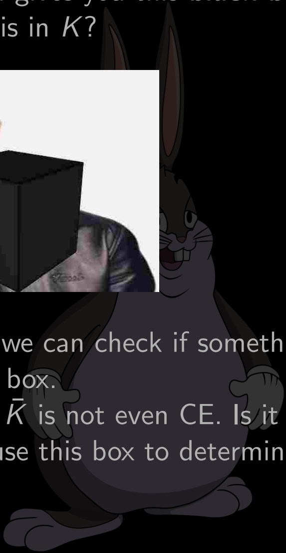

Is K now computable? Yes, because now we can check if something is in K or not by just feeding it into this black box.
What about $\bar{K}$? Without Eminem's help, $\bar{K}$ is not even CE. Is it now computable? Yes, because we can again use this box to determine if something is in $\bar{K}$ (so not in K) or not.

Turing reductions!

If we can compute (the indicator of) K, then we can compute $\bar{K}$. So in some sense, K is *at least as hard to compute* as K: once we are able to compute K, we will also be able to compute $\bar{K}$. We can *reduce* the problem of computing $\bar{K}$ to the problem of computing K.

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You may think of $A \leq_T B$ as saying "A is less difficult than B", in that we can reduce the problem of computing A into the problem of computing B.

Turing reductions!

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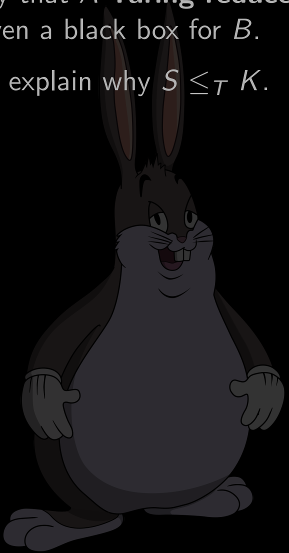
Task: Let S be a computable set. Briefly explain why $S \leq_T K$.



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Task: Let S be a computable set. Briefly explain why $S \leq_T K$.
Ans: Since S is computable, given a black box for K, we can just throw away the black box and compute S directly!

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Task: Let $K = \{x : \phi_x(x) \text{ halts}\}$, and

$$H = \{(x, e) : \phi_e(x) \text{ halts}\}.$$

Show that $K \leq_T H$.

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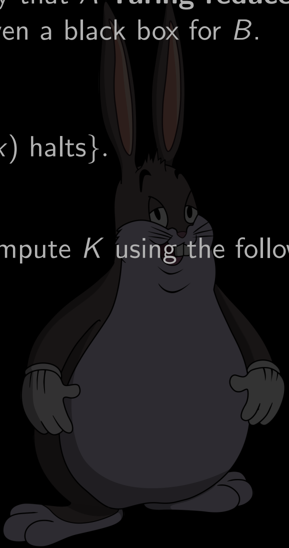
Show that $K \leq_T H$.
Ans: Given a black box for H, we can compute K using the following procedure:

```
def is_in_K(x):
  if (x, x) in H:
    return True
  else: return False
```



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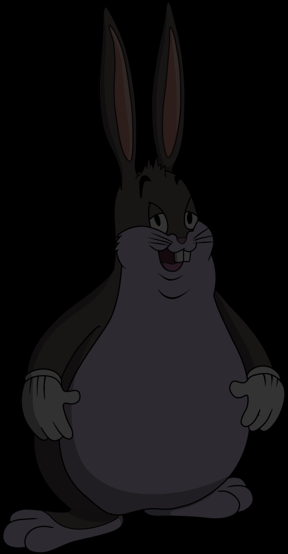

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Ans: Given a black box for K, we can compute H using the following procedure:

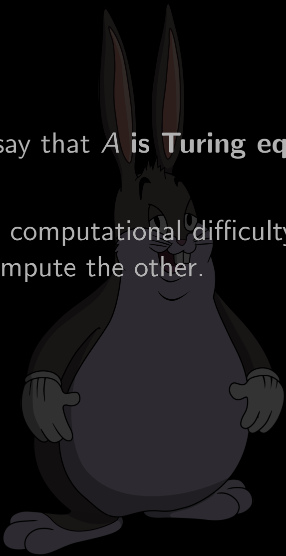
```
def is_in_H(x, e):
  Construct the TM M that does the following:
    M(y):
      (ignore y)
      Run the eth Turing machine on x
      Return if it halts
  Let z be the Turing Machine # of M
  if z in K:
    return True
  else: return False
```

Notice: We construct M, but we don't actually run it! Running M might result in a loop.

Turing reductions!

Definition: If $A \leq_T B$ and $B \leq_T A$, we say that A **is Turing equivalent to** B, and write $A =_T B$.

In some sense, this says A is equivalent in computational difficulty to B: if we can compute one, then we can also compute the other.

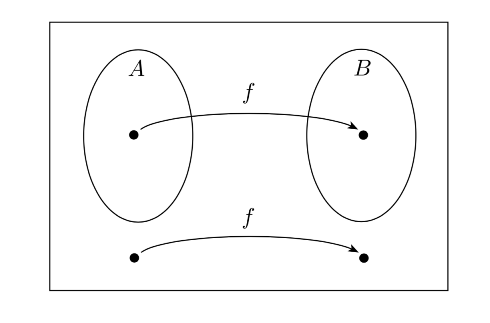


m-reductions...

We'll introduce another reduction mechanism, called an *m*-reduction.

Definition: Let A, B be sets. We say that $A \leq_m B$ if there exists a *computable* function f such that

$$x \in A \Leftrightarrow f(x) \in B.$$



stolen from `https://liyanxu.blog/2019/05/06/mapping-reducibility-turing-reducibility-kolmogorov-complexity/`

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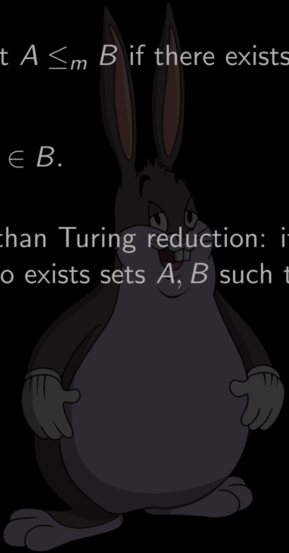
It turns out that *m*-reduction is stronger than Turing reduction: if $A \leq_m B$, then $A \leq_T B$. However, there do exists sets A, B such that $A \leq_T B$ but not $A \leq_m B$.



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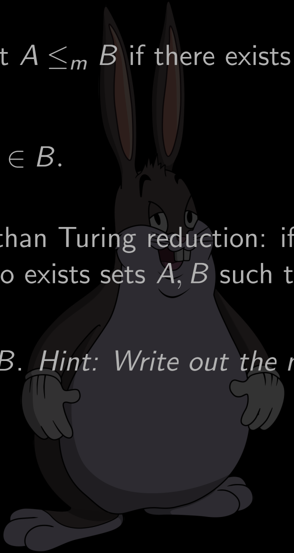
Task: Show that if $A \leq_m B$, then $A \leq_T B$. *Hint: Write out the meaning of each of those definitions.*



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Task: Show that if $A \leq_m B$, then $A \leq_T B$. *Hint: Write out the meaning of each of those definitions.*

Ans: Suppose $A \leq_m B$. Then there exists a computable function f such that

$$x \in A \Leftrightarrow f(x) \in B.$$

To show $A \leq_T B$, suppose we are given a black box for B. We can compute A as follows:

```
def is_in_A(x):
  return True if f(x) in B, False otherwise.
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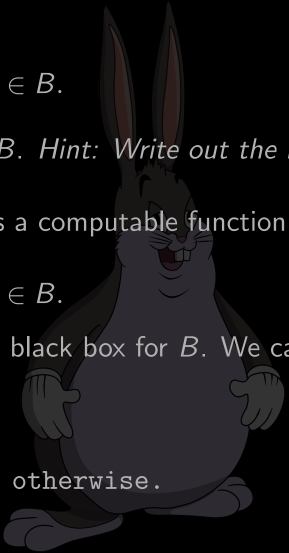

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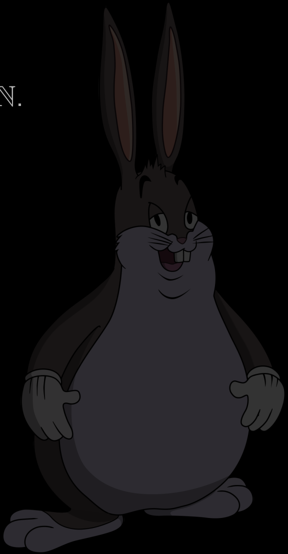

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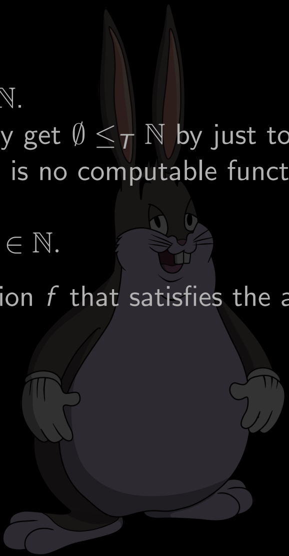
Task: Show that $\emptyset \leq_T \mathbb{N}$, but not $\emptyset \leq_m \mathbb{N}$.
Ans: $\emptyset$ is computable, so we automatically get $\emptyset \leq_T \mathbb{N}$ by just tossing away the black box for $\mathbb{N}$. However, there is no computable function f such that

$$x \in \emptyset \Leftrightarrow f(x) \in \mathbb{N}.$$

This is because there isn't even any function f that satisfies the above, regardless of computability of f! (Why?)



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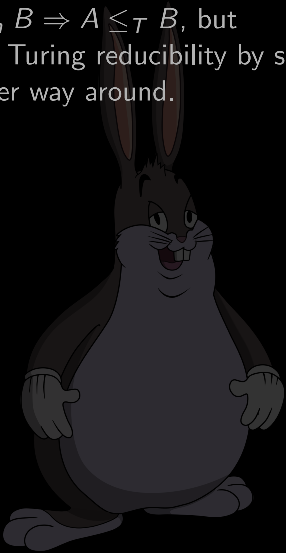
So what we have just shown is that $A \leq_m B \Rightarrow A \leq_T B$, but $A \leq_T B \not\Rightarrow A \leq_m B$. Thus, we may show Turing reducibility by showing m-reducibility, but not necessarily the other way around.



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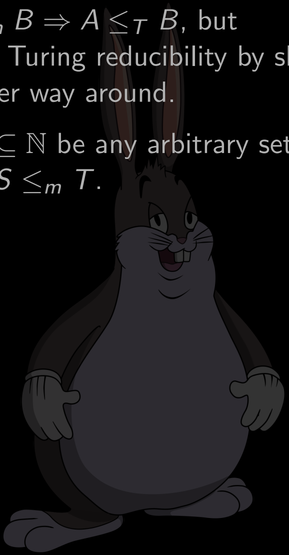
Task: Let $S \subseteq \mathbb{N}$ be computable, and $T \subseteq \mathbb{N}$ be any arbitrary set satisfying $T \neq \emptyset$ and $T \neq \mathbb{N}$. Show that $S \leq_m T$.



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Ans: Since $T \neq \emptyset$, there is some $p \in T$. Since $T \neq \mathbb{N}$, there is some $q \in \mathbb{N}, q \notin T$. Define f by

$$f(x) = \begin{cases} p & x \in S \\ q & x \notin S. \end{cases}$$

Since S is computable, so is f. Furthermore, $x \in S \Leftrightarrow f(x) \in T$.