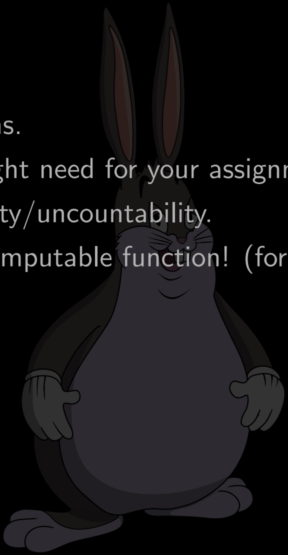
CSC363 Tutorial #5

“Review” (and some useful facts for A3?)

February 16, 2022

Learning objectives this tutorial

- Review *m*-reductions and 1-reductions.
- Prove some useful facts that you might need for your assignment...
- Recall/prove properties of countablility/uncountability.
- Show that "busy beaver" is a non-computable function! (for fun, of course)

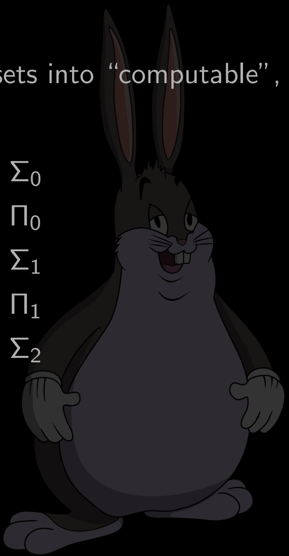


Stuff from like, the past few weeks

Task: Categorize these terms describing sets into “computable”, “CE”, or “neither”.

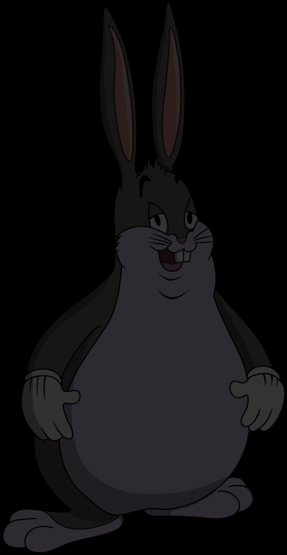
- Listable
- Diophantine
- Countable
- Uncountable
- Decidable

- Σ_0
- Π_0
- Σ_1
- Π_1
- Σ_2



Stuff from last lecture?

Question: What is the definition of $A \leq_m B$ and $A \equiv_1 B$?



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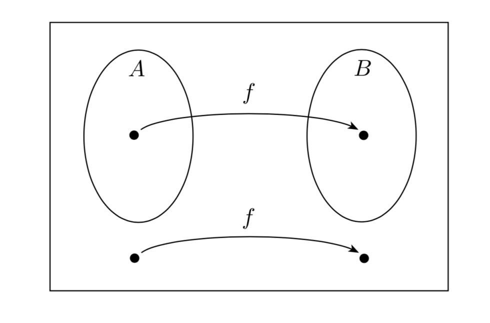
Question: What is the definition of $A \leq_m B$ and $A \equiv_1 B$?

Ans: Given two sets $A, B \subseteq \mathbb{N}$, we say that $A \leq_m B$ if there exists a *computable* $f : \mathbb{N} \to \mathbb{N}$ such that

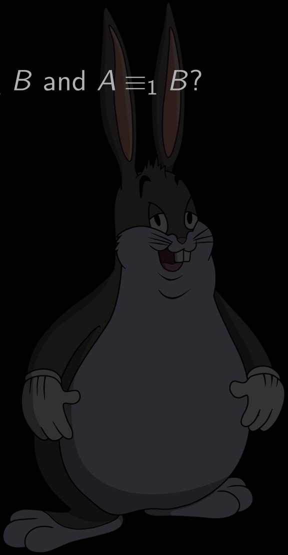
$$x \in A \Leftrightarrow f(x) \in B.$$



If $A \leq_m B$ and $B \leq_m A$, we say that $A \equiv_m B$.

Stuff from last lecture?

Question: What is the definition of $A \leq_1 B$ and $A \equiv_1 B$?



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Question: What is the definition of $A \leq_1 B$ and $A \equiv_1 B$?
Ans: Given two sets $A, B \subseteq \mathbb{N}$, we say that $A \leq_m B$ if there exists a *computable and injective* $f : \mathbb{N} \to \mathbb{N}$ such that

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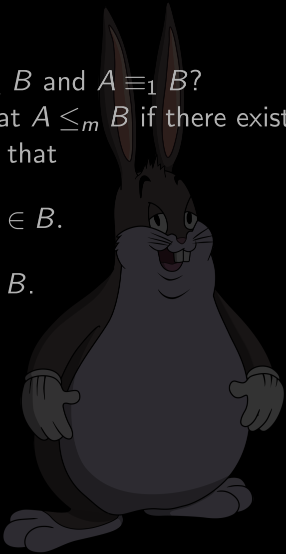

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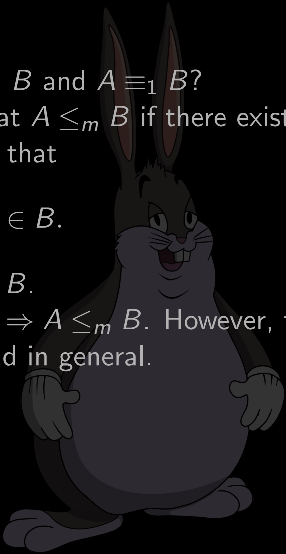
$$x \in A \Leftrightarrow f(x) \in B.$$

If $A \leq_1 B$ and $B \leq_1 A$, we say that $A \equiv_1 B$.
Immediately from the definitions, $A \leq_1 B \Rightarrow A \leq_m B$. However, the converse $A \leq_m B \Rightarrow A \leq_1 B$ does not hold in general.

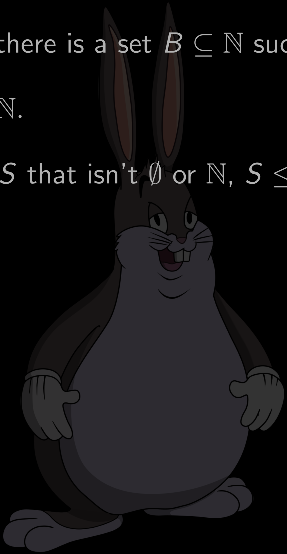


Assignment 3

Definition: A set $S \subseteq \mathbb{N}$ is a **cylinder** if there is a set $B \subseteq \mathbb{N}$ such that

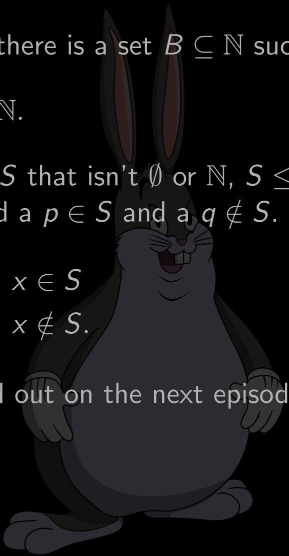
$$S \equiv_1 B \times \mathbb{N}.$$

Task: Show that for any computable set S that isn't $\emptyset$ or $\mathbb{N}$, $S \leq_1 S \times \mathbb{N}$.



[1] Or you may just prove it yourself instead.

Assignment 3



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Task: Show that for any computable set S that isn't $\emptyset$ or $\mathbb{N}$, $S \leq_1 S \times \mathbb{N}$.
Ans: Since $S \neq \emptyset$ and $S \neq \mathbb{N}$, we can find a $p \in S$ and a $q \notin S$. Define

$$f(x) = \begin{cases} \langle p, x \rangle & x \in S \\ \langle q, x \rangle & x \notin S. \end{cases}$$

f is computable and injective. Why? Find out on the next episode.[1]

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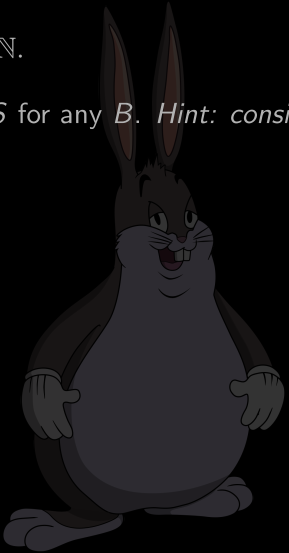

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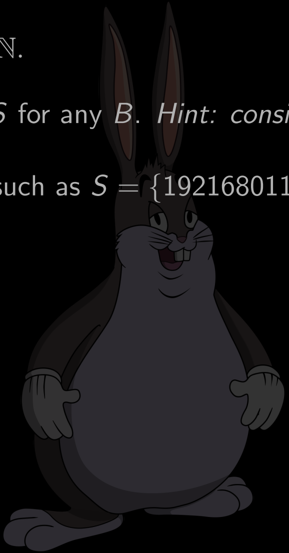
Task: Find a set S such that $B \times \mathbb{N} \not\leq_1 S$ for any B. *Hint: consider the cardinality of S.*
Ans: Any nonempty finite set S will do, such as $S = \{192168011\}$.
Consider any set B:



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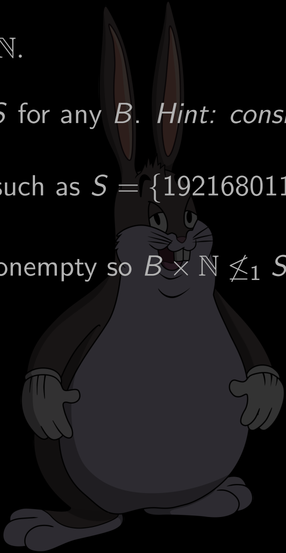

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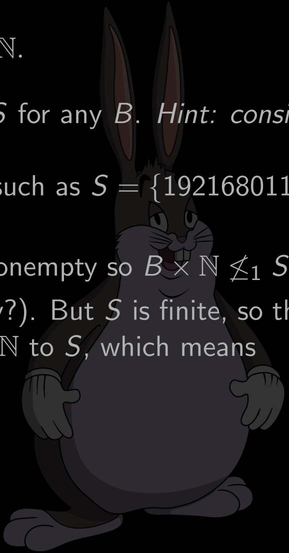
- If $B \neq \emptyset$, then $B \times \mathbb{N}$ is infinite (Why?). But S is finite, so there is no way to create an injection from $B \times \mathbb{N}$ to S, which means $B \times \mathbb{N} \not\leq_1 S$.



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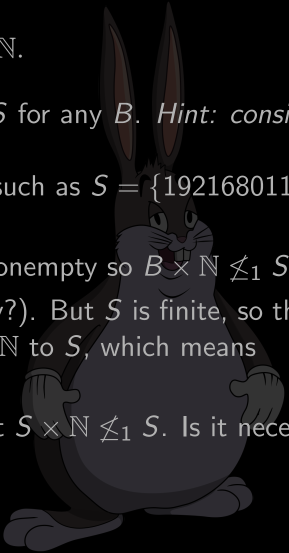
Question: We've found a set S such that $S \times \mathbb{N} \not\leq_1 S$. Is it necessarily true that for any set S, $S \times \mathbb{N} \leq_m S$?



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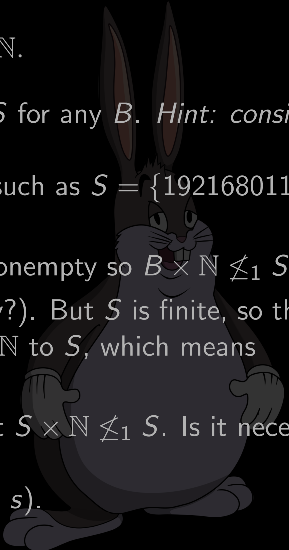

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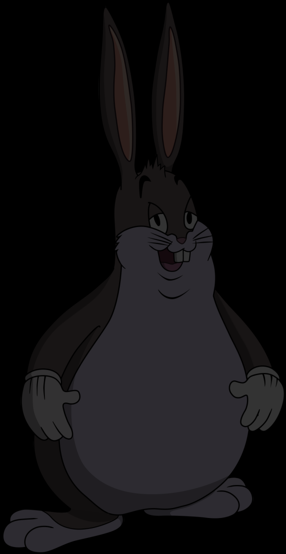
Question: We've found a set S such that $S \times \mathbb{N} \not\leq_1 S$. Is it necessarily true that for any set S, $S \times \mathbb{N} \leq_m S$?
Ans: Yes (consider the function $f\langle s, n\rangle = s$).

Speaking of cardinality...

Hope you remember your MAT102 stuff!



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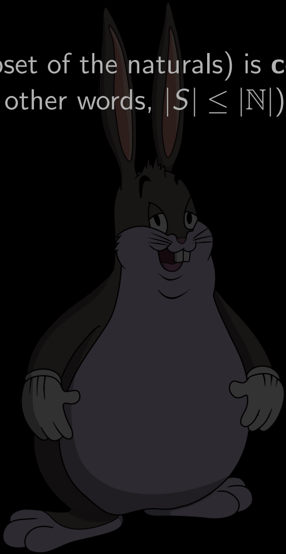
Definition: A set S (not necessarily a subset of the naturals) is **countable** if there is an injection from S to $\mathbb{N}$ (or in other words, $|S| \leq |\mathbb{N}|$).



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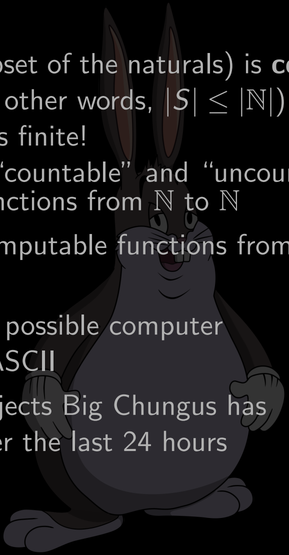
Note: “Countable” in this course includes finite!
Task: Categorize the following sets into “countable” and “uncountable”.

- $\mathbb{N}$
- $\mathbb{R}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{Q}$
- $\mathbb{N}^n$
- $P(\mathbb{N})$ (power set)
- The set of functions from $\mathbb{N}$ to $\mathbb{N}$
- The set of computable functions from $\mathbb{N}$ to $\mathbb{N}$
- The set of all possible computer programs in ASCII
- The set of objects Big Chungus has consumed over the last 24 hours



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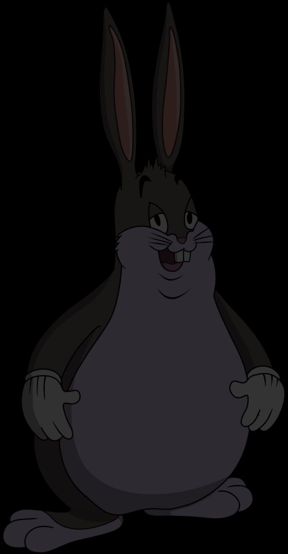

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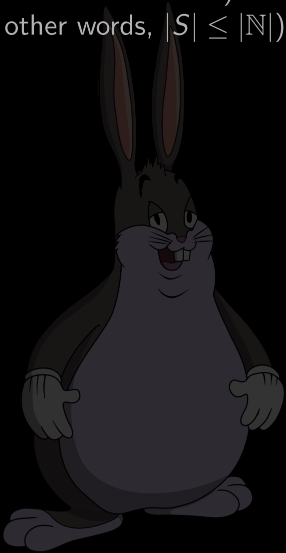

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Task: Let $S_0, S_1, \ldots$ be a countably infinite collection of countably infinite and pairwise disjoint sets. Show that

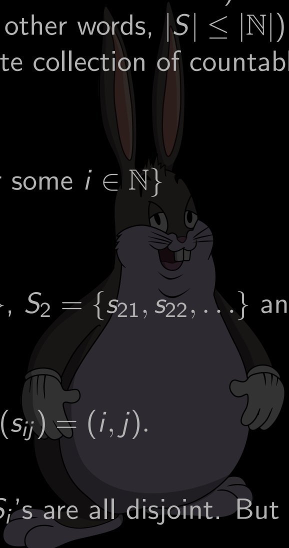
$$\bigcup_{i=0}^{\infty} S_i = \{s : s \in S_i \text{ for some } i \in \mathbb{N}\}$$

is countable.

Ans: Write it out! Let $S_1 = \{s_{11}, s_{12}, \ldots\}$, $S_2 = \{s_{21}, s_{22}, \ldots\}$ and so on. We can then define the injection

$$f : \bigcup_{i=0}^{\infty} S_i \to \mathbb{N} \times \mathbb{N}, f(s_{ij}) = (i, j).$$

This is well-defined and injective, as the S_i's are all disjoint. But we know $\mathbb{N} \times \mathbb{N}$ is countable already!



BusyBeaver

Not to be confused with the Android game *Busy Beaver*...

Here's the BusyBeaver(n) game!

> *Design a Turing machine with tape alphabet* $\{\square, 1\}$ *(*$\square$ *being the blank symbol), with* **exactly** n **states** *excluding the halting state. This Turing machine* **must halt** *with the empty string as input (starting will all* $\square$ *on the tape). Write as many '1' symbols as you can to the tape, before halting.*

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Task: Play BusyBeaver(1). That is, build a 1-state (excluding the halting state) TM that writes as many '1's to the tape as possible before halting.

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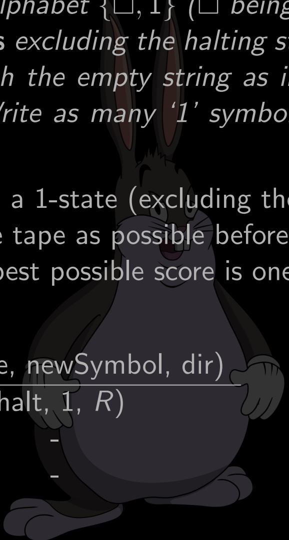
Task: Play BusyBeaver(1). That is, build a 1-state (excluding the halting state) TM that writes as many '1's to the tape as possible before halting.
Ans: There isn't much we can do... the best possible score is one '1' symbol.

State	Symbol	(newState, newSymbol, dir)
start	$\square$	(halt, 1, R)
start	1	-
halt	-	-



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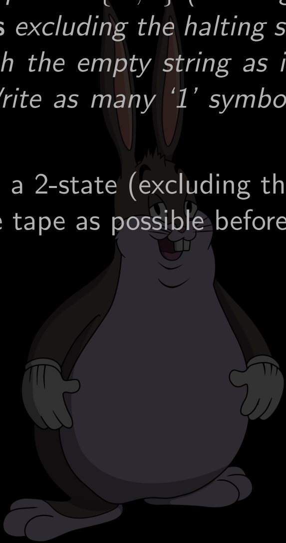
Task: Play BusyBeaver(2). That is, build a 2-state (excluding the halting state) TM that writes as many '1's to the tape as possible before halting.



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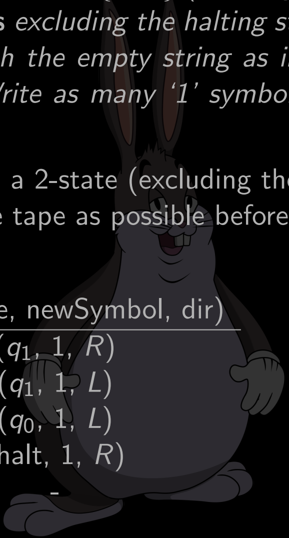
Task: Play BusyBeaver(2). That is, build a 2-state (excluding the halting state) TM that writes as many '1's to the tape as possible before halting.
Ans: The best possible score is 4.

State	Symbol	(newState, newSymbol, dir)
start	$\square$	(q_1, 1, R)
start	1	(q_1, 1, L)
q_1	$\square$	(q_0, 1, L)
q_1	1	(halt, 1, R)
halt	-	-



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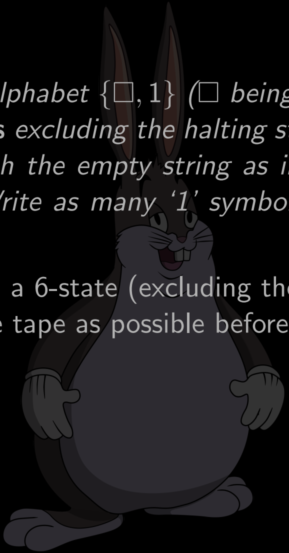
Task: Play BusyBeaver(6). That is, build a 6-state (excluding the halting state) TM that writes as many '1's to the tape as possible before halting.



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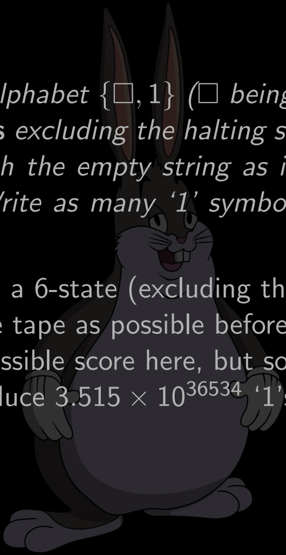

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Task: Play BusyBeaver(6). That is, build a 6-state (excluding the halting state) TM that writes as many '1's to the tape as possible before halting.
Ans: We actually don't know the best possible score here, but someone came up with a 6-state TM that can produce 3.515×10^{36534} '1's.

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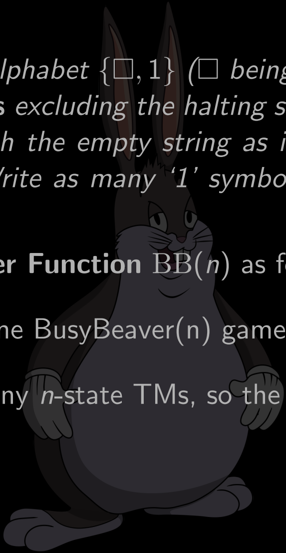
Of course, we can define the **Busy Beaver Function** $\mathrm{BB}(n)$ as follows:

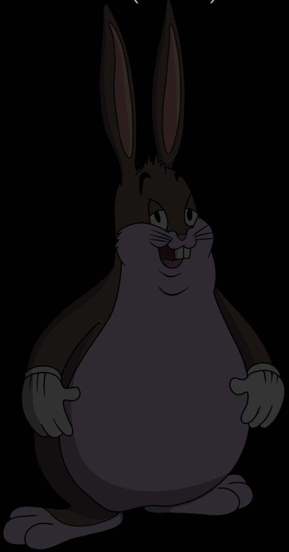
$$\mathrm{BB}(n) = \text{Best possible score in the BusyBeaver(n) game.}$$

For each $n \in \mathbb{N}$, there are only finitely many n-state TMs, so the maximum possible score exists.



BusyBeaver

Task: Prove that $BB(n)$ is not a computable function. That is, there is no program that can compute $BB(n)$ for all $n \in \mathbb{N}$. (Hard!)



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Ans: Suppose there were such a program `compute_bb(n)`, towards a contradiction. Build the following program UTM:

```
UTM(e, x):
  Create two tapes tape1 and tape2.
  While True:
    On tape1:
      Run one step of the eth turing machine on x.
    On tape2, write a 1.
    If the eth TM halted, erase tape1, and return.
```

We can build a Turing machine that runs UTM, say with k states. But then we can write the following program:

```
H(e, x):
  max_steps = compute_bb(k)
  Run UTM(e, x) while the # of 1s on tape2 < max_steps.
  If it halts, return True. Else, return False.
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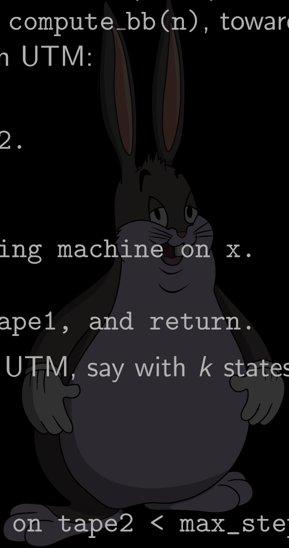

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```
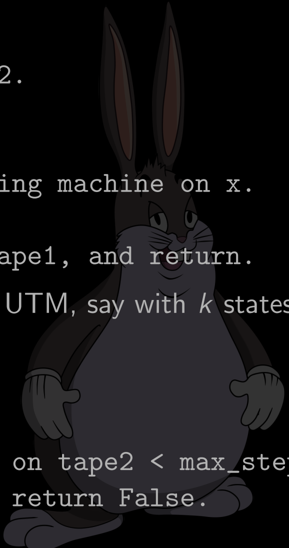
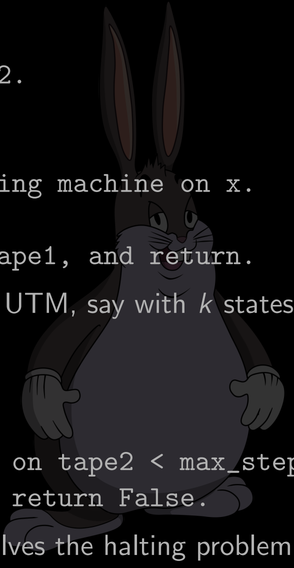

Task: Spot the contradiction.



BusyBeaver

Ans: Suppose there were such a program compute_bb(n), towards a contradiction. Build the following program UTM:

```
UTM(e, x):
  Create two tapes tape1 and tape2.
  While True:
    On tape1:
      Run one step of the eth turing machine on x.
    On tape2, write a 1.
    If the eth TM halted, erase tape1, and return.
```



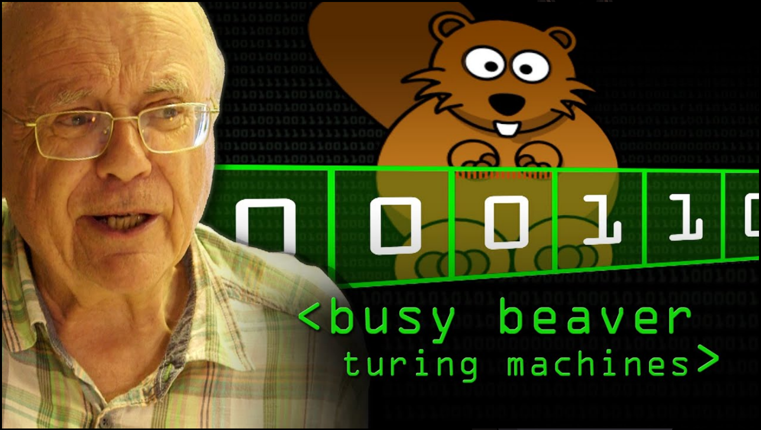
We can build a Turing machine that runs UTM, say with k states. But then we can write the following program:

```
H(e, x):
  max_steps = compute_bb(k)
  Run UTM(e, x) while the # of 1s on tape2 < max_steps.
  If it halts, return True. Else, return False.
```

Task: Spot the contradiction. **Ans:** H solves the halting problem!

BusyBeaver

Some video on BusyBeaver by Computerphile:
`https://www.youtube.com/watch?v=CE8UhcyJS0I`



You get to listen to this guy talk about BB!