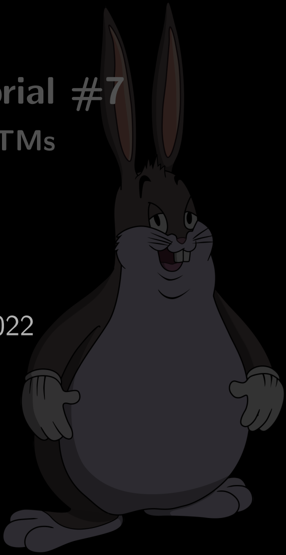
CSC363 Tutorial #7

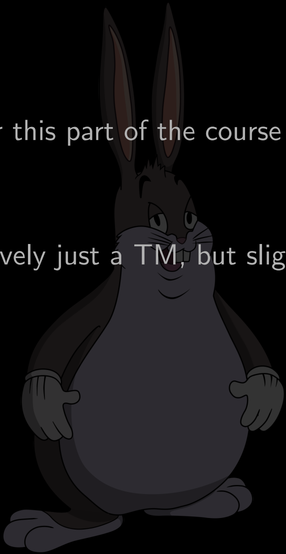
Alternative TMs



March 9, 2022

Learning objectives this tutorial

- Define some aliases we'll be using for this part of the course (complexity).
- Describe a "multi-tape" TM.
- Show that a multi-tape TM is effectively just a TM, but slightly better.



Some aliases

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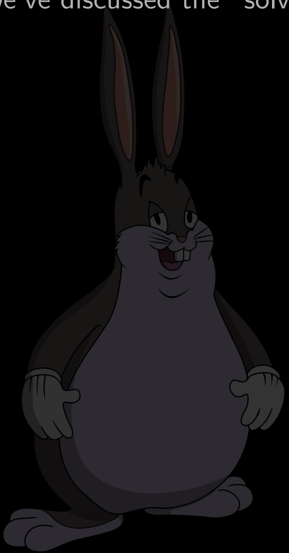

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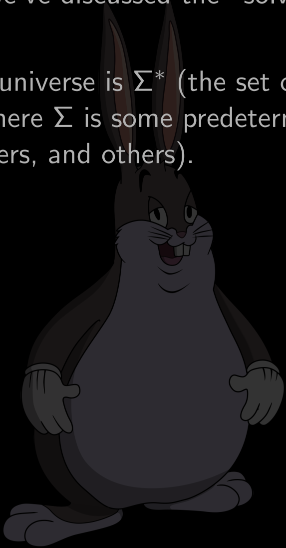

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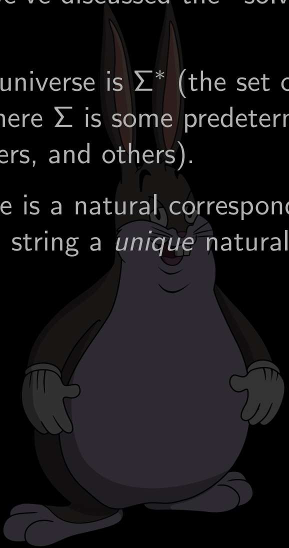

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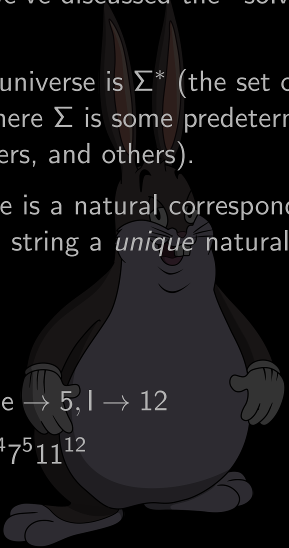
Ans: Gödel Numbers!

$$\mathrm{g} \to 7, \mathrm{o} \to 15, \mathrm{d} \to 4, \mathrm{e} \to 5, \mathrm{l} \to 12$$

$$\mathrm{godel} \to 2^7 3^{15} 5^4 7^5 11^{12}$$



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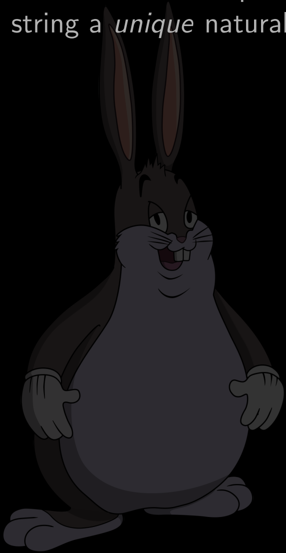

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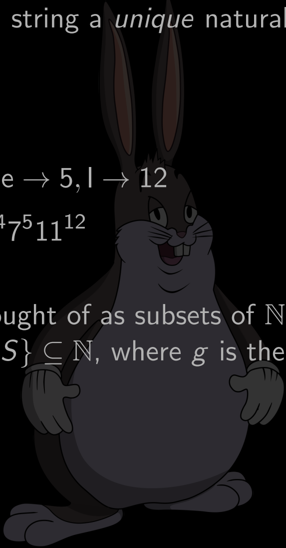

So in this sense, subsets of Σ^* can be thought of as subsets of $\mathbb{N}$ by mapping $S \subseteq \Sigma^*$ to $g(S) = \{g(w) : w \in S\} \subseteq \mathbb{N}$, where g is the Gödel mapping function.

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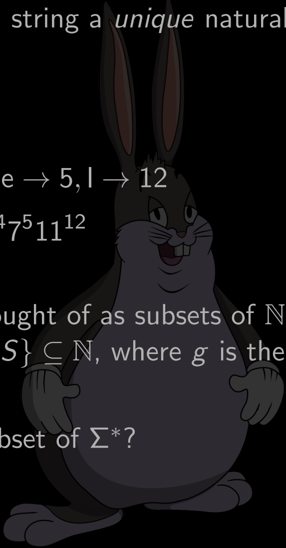

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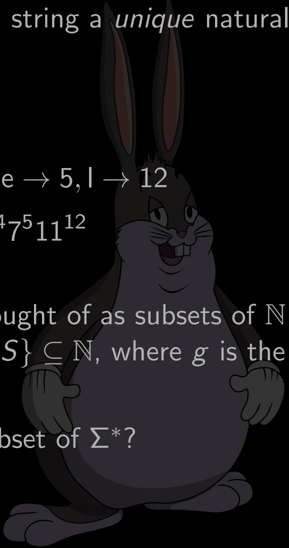

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Question: What is another term for a subset of Σ^*?
Ans: *Language*.

Some aliases

In complexity, we will consider the efficient *decidability* of languages.

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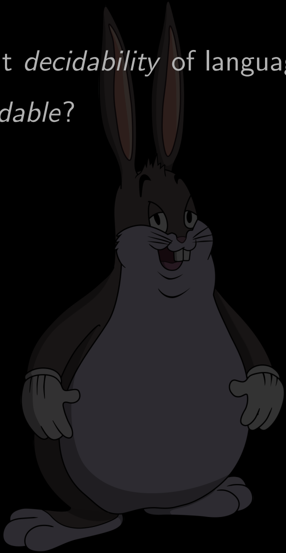

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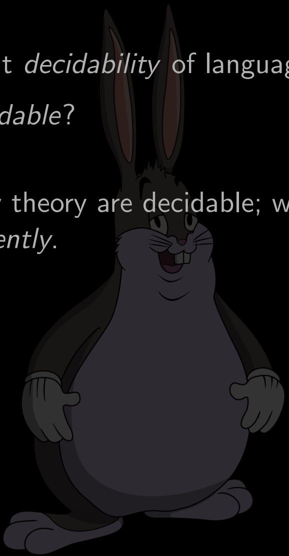

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Most languages we consider in complexity theory are decidable; we examine whether they are decidable *efficiently*.

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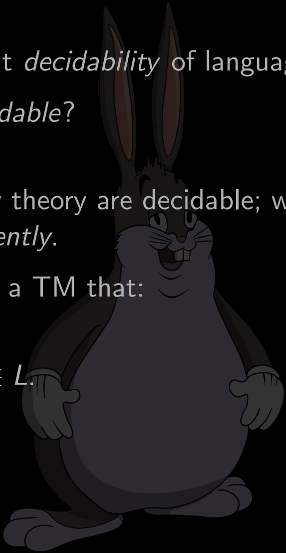
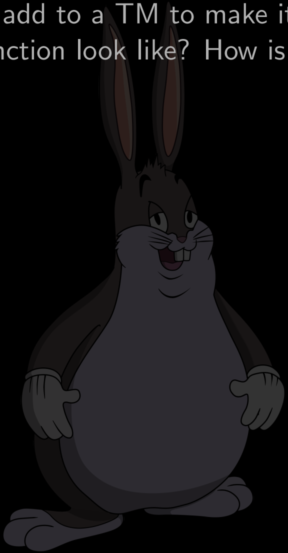
Definition: A *decider* for a language L is a TM that:

1. always halts *on any input*,
2. accepts the input x if and only if $x \in L$.



Multi-tape TM

Task: Describe some attributes we could add to a TM to make it "multi-tape". What will the transition function look like? How is input handled?



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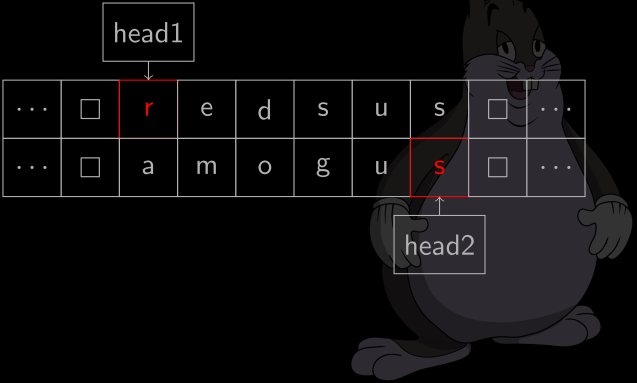
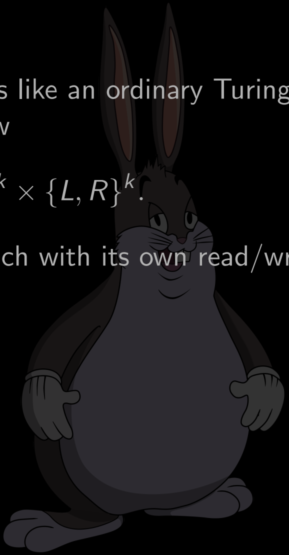


Multi-tape TM

Definition: A *k*-**tape Turing Machine** is like an ordinary Turing Machine, but its transition function is now

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k.$$

In effect, this gives us *k* distinct tapes, each with its own read/write head. We read and write *k* symbols at once.



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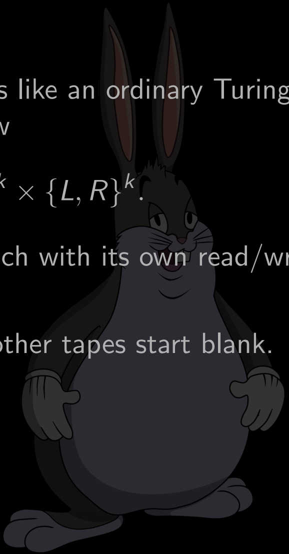
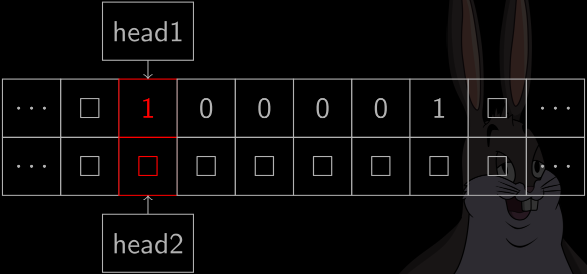
The input is placed on the first tape; all other tapes start blank.



Multi-tape TM Example

Let's construct a multi-tape TM over the alphabet $\{0, 1\}$ that accepts palindromes. Here's what we will do:



1. Copy the string on tape 1 to tape 2.
2. Move head1 to the beginning of the first tape.
3. Compare characters from head1 and head2, scanning right and left respectively.

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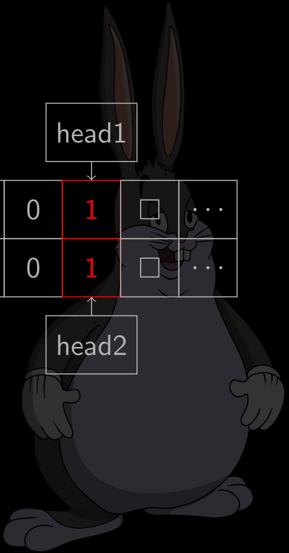
⋯	□	1	0	0	0	0	1	□	⋯
⋯	□	1	0	0	0	0	1	□	⋯

head2



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3. Compare characters from head1 and head2, scanning right and left respectively.

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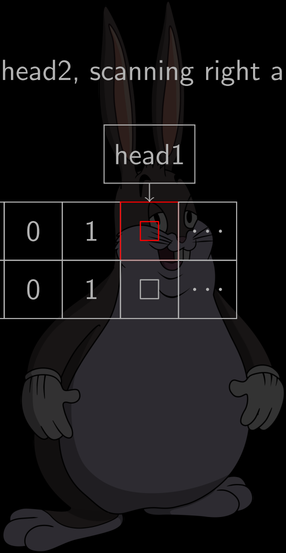

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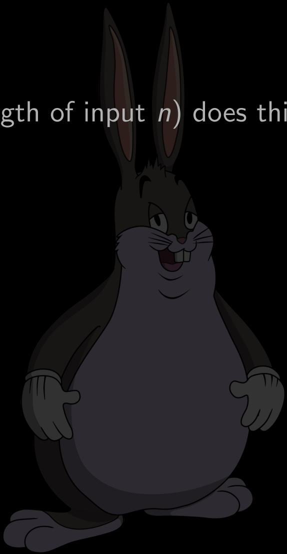
Question: What runtime (in terms of length of input n) does this 2-tape TM have?



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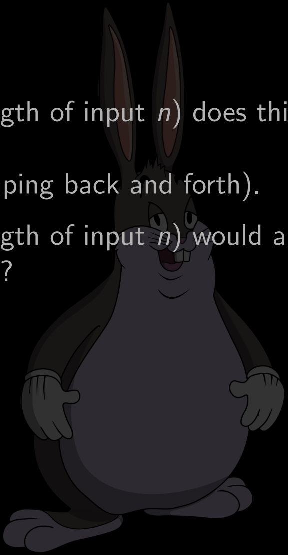

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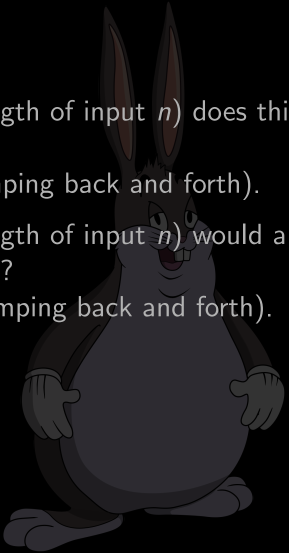
Ans: $O(n^2)$ (because we have to keep jumping back and forth).



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Task: Construct a $O(n)$ 2-tape TM that decides the language $\{0^n 1^n : k \in \mathbb{N}\}$. You may use a high-level description if you want.

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Ans: Here's the procedure I have in mind.

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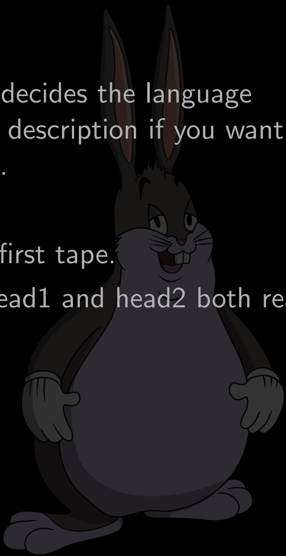

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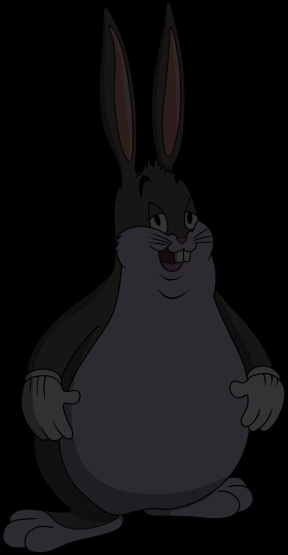
Question: How fast can we decide $\{0^n 1^n : k \in \mathbb{N}\}$ with a TM?



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Question: How fast can we decide $\{0^n1^n : k \in \mathbb{N}\}$ with a TM?
Ans: Naively, $O(n^2)$. The procedure is as follows:

1. Cross out a 0; move to the right end of the string.
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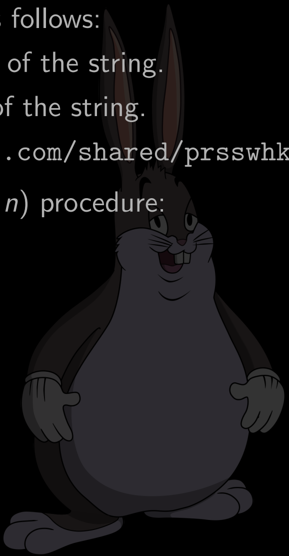

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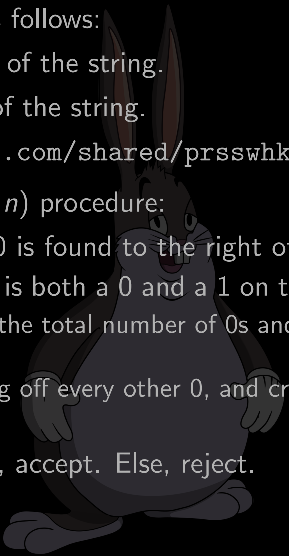
1. Scan across the tape and reject if a 0 is found to the right of a 1.
2. Repeat the following as long as there is both a 0 and a 1 on the tape:
   2.1 Scan across the tape, and reject if the total number of 0s and 1s remaining is odd.
   2.2 Scan again across the tape, crossing off every other 0, and crossing off every other 1.
3. If the tape doesn't have any 0s or 1s, accept. Else, reject.



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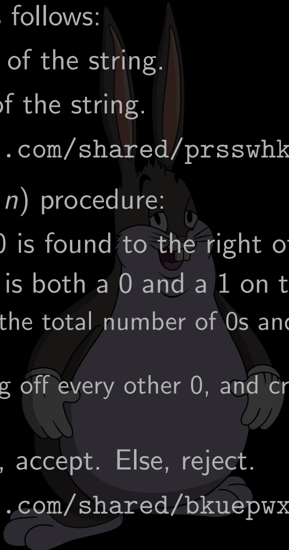

Multi-tape TMs in P

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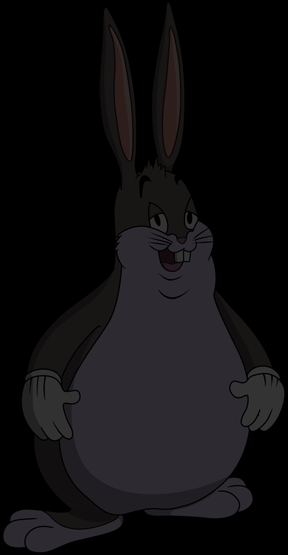

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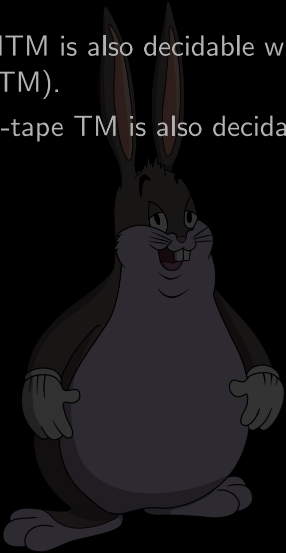
- Everything that is decidable with a k-tape TM is also decidable with a TM.



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Theorem: Everything that is decidable with a k-tape TM in $O(f(n))$ time is decidable with a TM in $O((f(n))^2)$ time. (See Sipser page 137)

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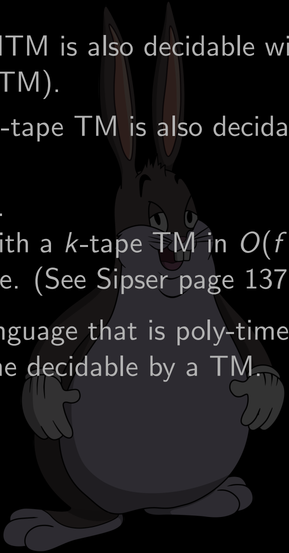
Task: Using the above, show that any language that is poly-time decidable by a k-tape TM is also poly-time decidable by a TM.



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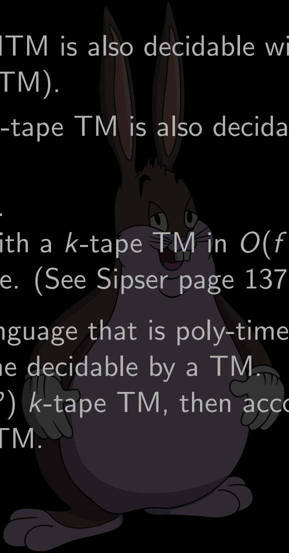

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In effect, this shows that multi-tape TMs are "better", but don't fundamentally change the set of poly-time decidable languages.