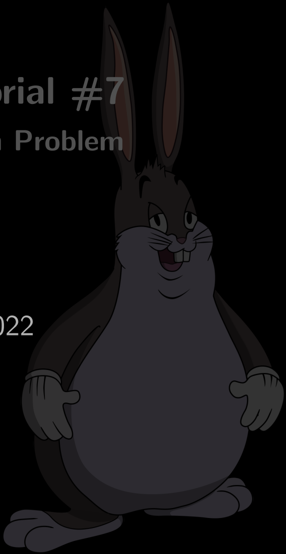
CSC363 Tutorial #7

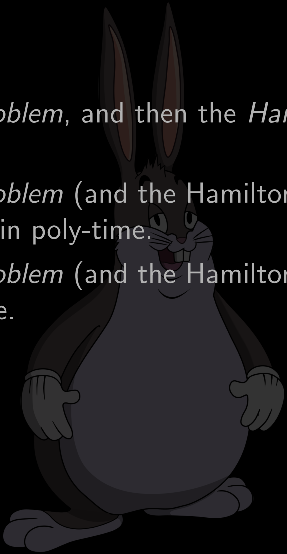
Hamiltonian Path Problem

March 9, 2022

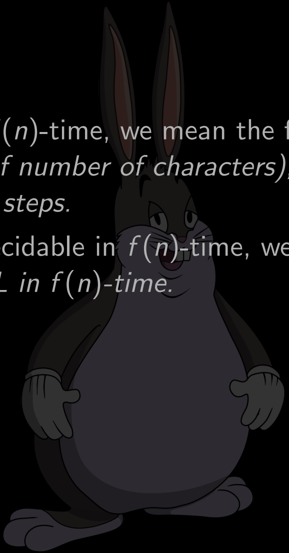


Learning objectives this tutorial

- Formulate the *Hamiltonian Cycle Problem*, and then the *Hamiltonian Path Problem*.
- Show that the *Hamiltonian Cycle Problem* (and the Hamiltonian Path Problem) can be decided by a NTM in poly-time.
- Show that the *Hamiltonian Cycle Problem* (and the Hamiltonian Path Problem) can be *verified* in poly-time.



Some Clarifications



- When we say that a TM M runs in $f(n)$-time, we mean the following: *For all inputs of length n (in terms of number of characters), the computation $M(n)$ halts within $f(n)$ steps.*
- When we say that a language L is decidable in $f(n)$-time, we mean that *there is some TM that decides L in $f(n)$-time.*

Hamilton

Question: Who's this person?

Hamilton

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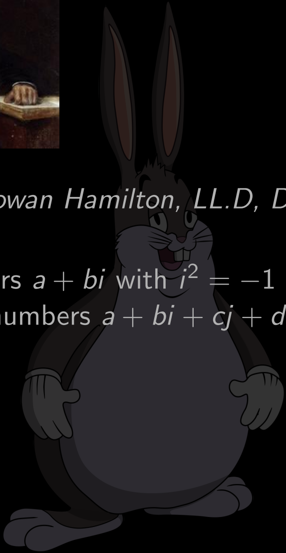
Ans: Sir William Rowan Hamilton, LL.D, DCL, MRIA, FRAS.

Hamilton Lore

List of things attributed to *Sir William Rowan Hamilton, LL.D, DCL, MRIA, FRAS*:

Hamilton Lore



List of things attributed to *Sir William Rowan Hamilton, LL.D, DCL, MRIA, FRAS*:

- Quaternions. (Think complex numbers $a + bi$ with $i^2 = -1$ aren't enough? Introducing 4-dimensional numbers $a + bi + cj + dk$ with $i^2 = j^2 = k^2 = ijk = -1$!)

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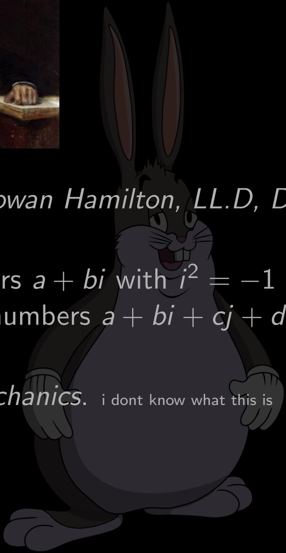

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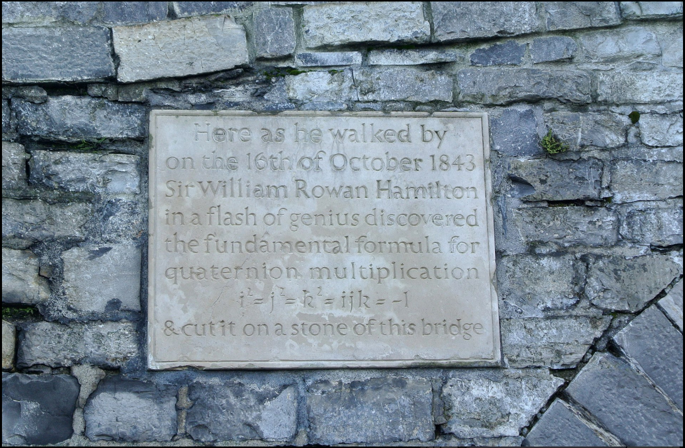
- Quaternions. (Think complex numbers $a + bi$ with $i^2 = -1$ aren't enough? Introducing 4-dimensional numbers $a + bi + cj + dk$ with $i^2 = j^2 = k^2 = ijk = -1$!)
- Physics, specifically *Hamiltonian Mechanics*. i dont know what this is
- Astronomy. i dont know astronomy either
- Some graph theory.
- Other uninteresting stuff

Hamilton Lore

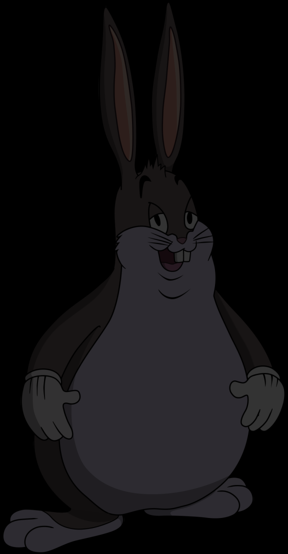


Around Broom Bridge, Dublin. Vandalized many times, of course.

There's also a *Hamilton Walk* event that takes place every year from Dunsink Observatory in Dublin Broom Bridge. Would be funny if they walked in a *Hamiltonian path*, eh.

Hamilton

Question: What would someone, living in 19th century Ireland, do when they were bored?



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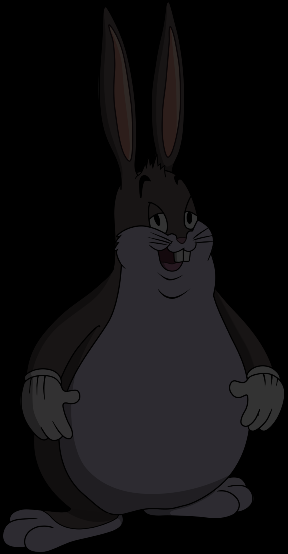

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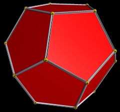

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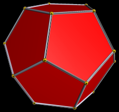
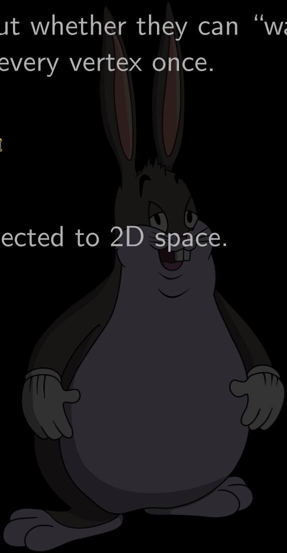


Let's try it! Here's the dodecahedron projected to 2D space.



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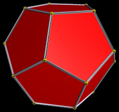
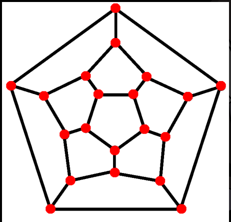


Let's try it! Here's the dodecahedron projected to 2D space.
Task: Can you find a cycle in the graph below that visits every vertex exactly once?



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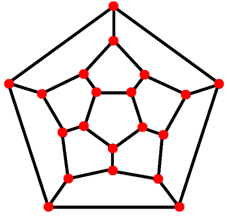
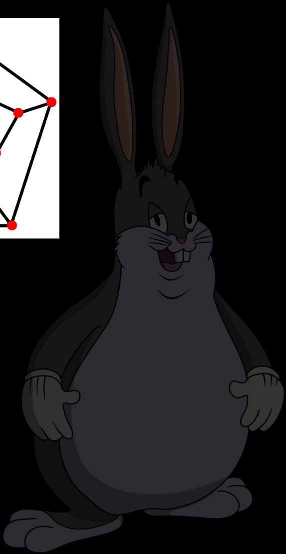
Task: Can you find a cycle in the graph below that visits every vertex exactly once?





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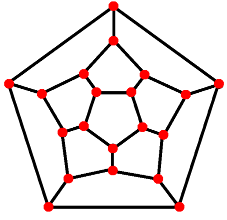
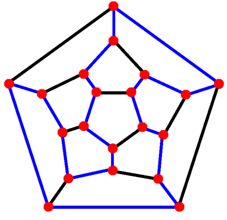
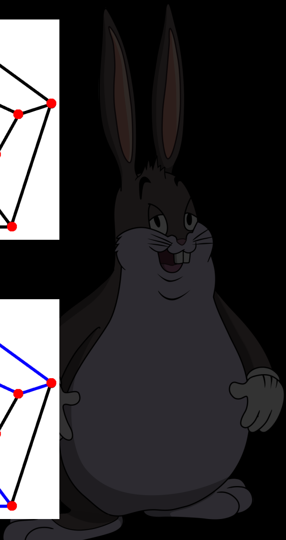


Ans: Yes!

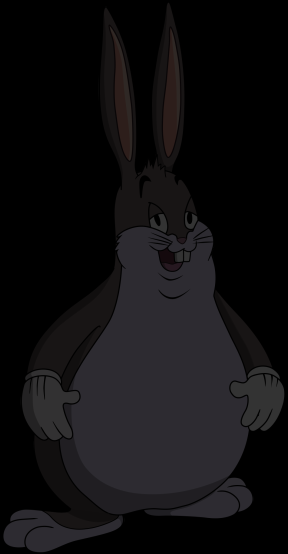




This kinda reminds me of "connect the dots" games I used to play as a kid

Hamiltonian Cycle Problem

The previous “connect the dots in a loop” game is a specific instance of the **Hamiltonian Cycle Problem**.



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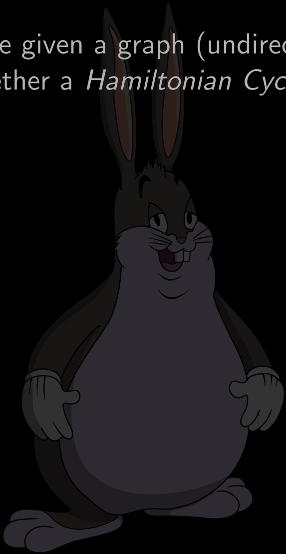
In the Hamiltonian Cycle Problem, you are given a graph (undirected in our context), and asked to determine whether a *Hamiltonian Cycle* exists.



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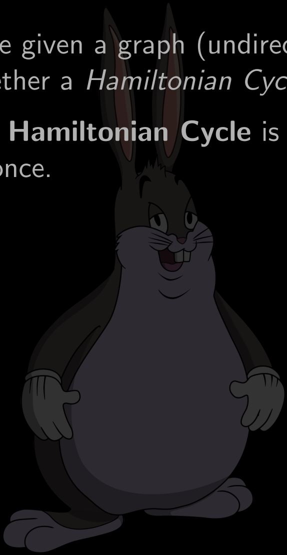
Definition: Given a graph $G = (V, E)$, a **Hamiltonian Cycle** is a cycle that contains each vertex $v \in V$ exactly once.



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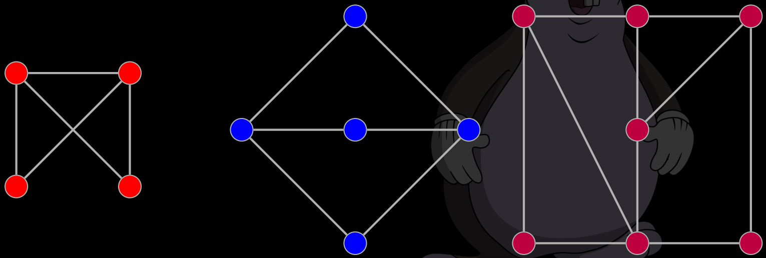
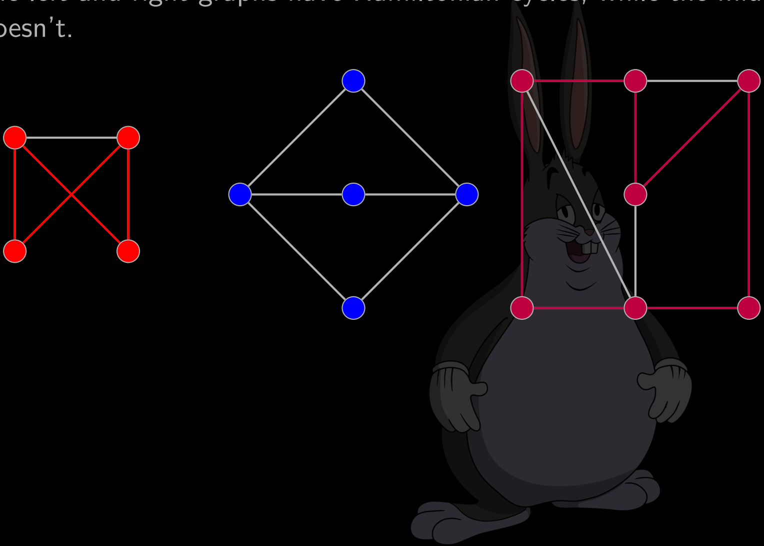
Task: Determine whether Hamiltonian cycles exist in the following graphs.



Hamiltonian Cycle Problem

Ans: The left and right graphs have Hamiltonian cycles, while the middle graph doesn't.



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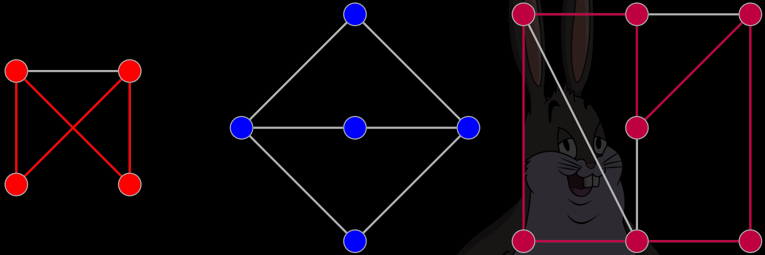

Question: How do we know that there is no Hamiltonian cycle for the middle graph?

Hamiltonian Cycle Problem

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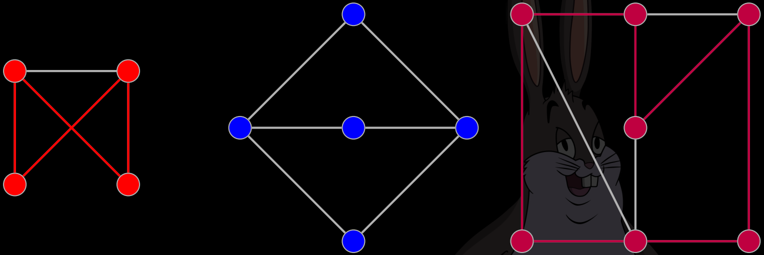

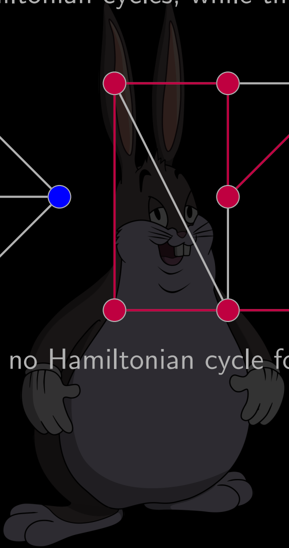

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Ans: 💀brute force.

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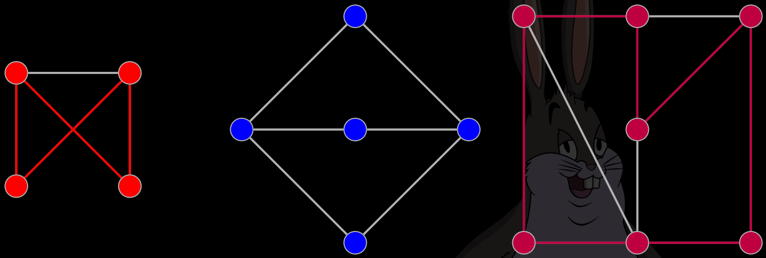

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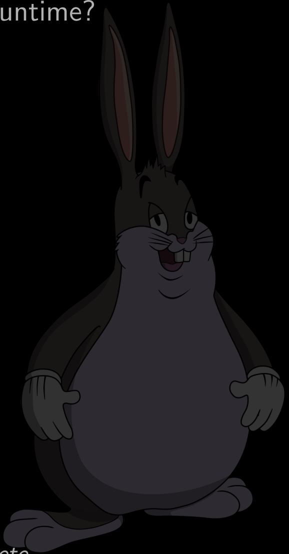
Insight: This gives us another "hard to solve, easy to verify" problem.

Hamiltonian Cycle Problem

Task: Describe, in pseudocode, how you can "brute-force" the Hamiltonian cycle problem. What is the runtime?



[2] The Hamiltonian Cycle problem is *NP-complete*.

Hamiltonian Cycle Problem

Task: Describe, in pseudocode, how you can "brute-force" the Hamiltonian cycle problem. What is the runtime?
Ans:

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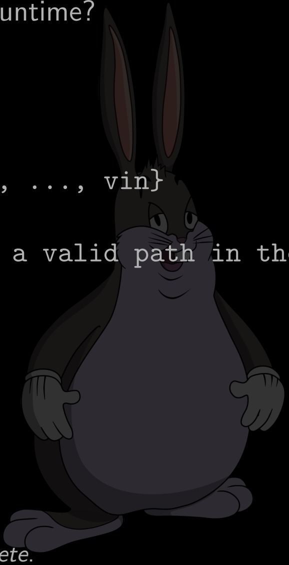
def has_hcycle(V, E):
  suppose V = {v1, v2, ..., vn}
  for every permutation {vi1, vi2, ..., vin}
    of {v1, v2, ..., vn}:
    if vi1->vi2->...->vin->vi1 is a valid path in the graph:
      return True
  return False
```

The runtime is...



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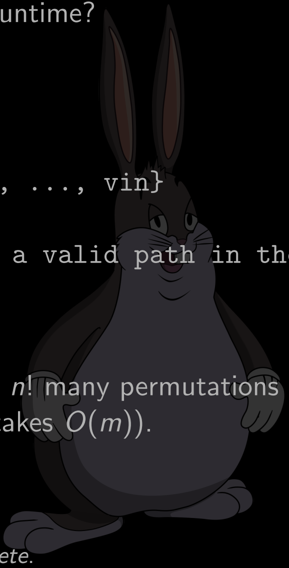
The runtime is... $O(n!m)$ (since there are $n!$ many permutations of n vertices, and checking each permutation takes $O(m)$).



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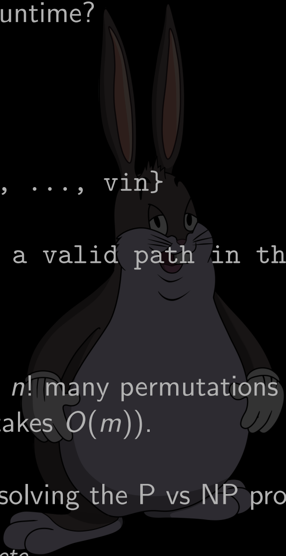

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Question: Can we do better?
Ans: 💀 we don't know... (this is akin to solving the P vs NP problem)[2]

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Hamiltonian Cycle Problem

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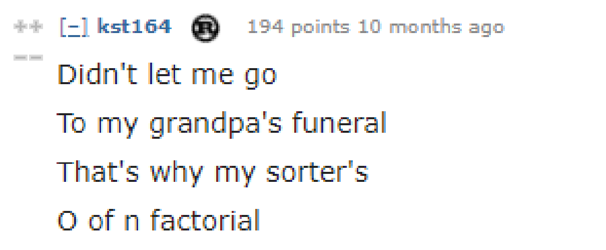
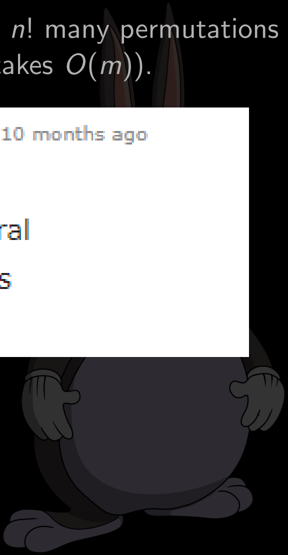

[–] kst164 194 points 10 months ago

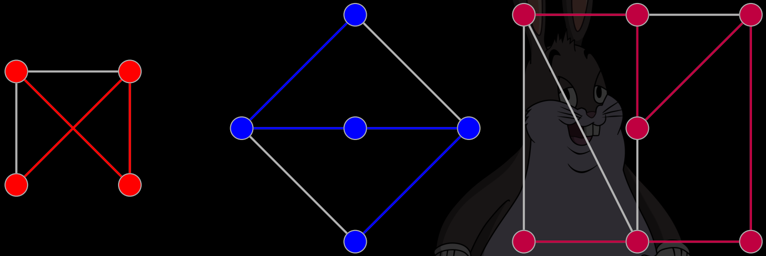
Didn't let me go
To my grandpa's funeral
That's why my sorter's
O of n factorial

source



Hamiltonian Cycle Problem

There's a similar problem, called the **Hamiltonian Path Problem**, which involves visiting every vertex exactly once in a path (without having to loop back to the beginning).



The middle graph has a solution to the Hamiltonian Path Problem.

NTM solution to Hamiltonian Cycle Problem

Task: Build a poly-time NTM that decides the language

$$\mathrm{HC} = \{G : \text{There is a Hamiltonian Cycle in } G\}.$$

You should define `in_HC(V, E)` (where (V, E) is the graph).

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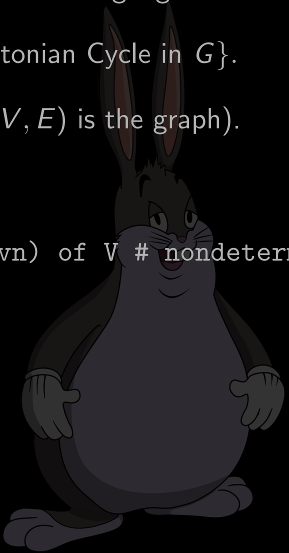
Ans:

```
in_HC(V, E):
  choose a permutation (v1, ..., vn) of V # nondeterministic!
  for i in 1, ..., n-1:
    if (vi, v(i+1)) not in E:
      reject
  if (vn, v1) not in E:
    reject
  accept
```



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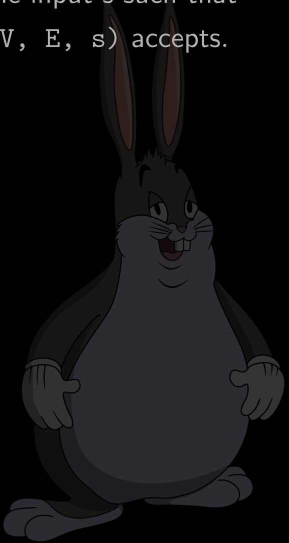
Task: Build a poly-time TM `verify_HC(V, E, s)` such that

$$(V, E) \in \mathrm{HC} \Leftrightarrow \text{There is some input s such that } \texttt{verify\_HC(V, E, s)} \text{ accepts.}$$



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Ans:

```
verify_HC(V, E, s: string):
  let V = (v1, ..., vn)
  if s is not of the form "(v{k1}, ..., v{kn}})":
    reject
  parse s to extract vertices vk1, ..., vkn
  for i in range(n-1):
    if (vki, vk(i+1)) not in E:
      reject
  if (vkn, vk1) not in E:
    reject
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```

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```

`verify_HC(V, E, s)` acts as a *verifier*: it checks whether a prospective "solution" `s` to the Hamiltonian Cycle problem actually works.

NTM poly-time versus verifiable in poly-time

There is a more general definition of a *verifier*.

Definition: A verifier V for a language L is a Turing machine that satisfies

$$x \in L \Leftrightarrow (\exists s)\text{V(x, s) accepts.}$$

A string s is called a **certificate** if $V(x, s)$ accepts.