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it's the last tutorial! :D

topics:
- equivalence relations
- equivalence classes
- modular arithmetic (hidden in the problems)

Definition 7.2.1. An **equivalence relation** R on a set S is a relation (that is, $R \subseteq S \times S$), such that:

(a) For any $x \in S$, $(x, x) \in R$ (**reflexive** property).

(b) For any $x, y \in S$, if $(x, y) \in R$, then $(y, x) \in R$ (**symmetric** property).

(c) For any $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ (**transitive** property).

1) (a) Prove directly, that the following relation, on the set of integers, is an equivalence relation.

$a \equiv b$ if and only if $a - b$ is divisible by 4.

- Reflexive: is $a \equiv a$ $\forall a \in \mathbb{Z}$?
  yes! $a - a = 0$ which is divisible by 4.

- Symmetric: does $a \equiv b \Rightarrow b \equiv a$?
  yes! if $a \equiv b$, then $a - b = 4k$ for some $k \in \mathbb{Z}$.
  $b - a = 4(-k)$.
  so $b - a$ is divisible by 4,
  so $b \equiv a$.

- Transitive: does $a \equiv b$, $b \equiv c \Rightarrow a \equiv c$?
  yes! if $a \equiv b$, $b \equiv c$
  $a - b = 4k$ $\quad b - c = 4l$ for some $k, l \in \mathbb{Z}$.
  $a - c = (a-b) + (b-c)$
  $= 4k + 4l = 4(k+l)$
  so $a - c$ is divisible by 4, so $a \equiv c$.

(b) Show that if two integers satisfy the relation in part (a), then they have the same remainder when divided by 4 (refer to Theorem 6.1.2 and Exercise 6.4.7).

Theorem 6.1.2. *(The Division Algorithm)*
If $a, b \in \mathbb{N}$, then there is a unique pair of integers, q and r, with $q \geq 0$ and $0 \leq r < b$, such that

$$a = q \cdot b + r$$

*(q is called the **quotient**, and r the **remainder**).*

6.4.7. There are ways to generalize the Division Algorithm (Theorem 6.1.2), so that it can be applied to all integers (both positive and negative). Here is one possible generalization.

If $a, b \in \mathbb{Z}$, with $b \neq 0$, then there is a unique pair of integers, q and r, with $0 \leq r < |b|$, such that $a = q \cdot b + r$.

Note that the remainder is still required to be nonnegative, so for instance, if we divide -21 by -4, the quotient is 6 and the remainder is 3, as $-21 = 6 \cdot (-4) + 3$.

(a) Find the quotient and the remainder, obtained when dividing a by b.

- $a = 27$ and $b = -8$.
- $a = -15$ and $b = 2$.
- $a = -36$ and $b = -9$.
- $a = 5$ and $b = -7$.
- $a = -4$ and $b = 9$.

(b) Prove the generalized version of the Division Algorithm given above.
(Hint: Instead of using induction, proceed by cases, and refer to Theorem 6.1.2.)

(you don't have to actually do 6.4.7 as an exercise)

$a \equiv b$ if and only if $a - b$ is divisible by 4.

Suppose $4 \mid a - b$.

by the division algorithm,
there exist unique integers q_a, r_a
and q_b, r_b such that

$a = q_a \cdot 4 + r_a$ $\qquad 0 \leq r_a, r_b < 4$

$b = q_b \cdot 4 + r_b$

show $r_a = r_b$:

$4 \mid a - b$

- $a = -36$ and $b = -9$.

(b) Prove the generalized version of the Division Algorithm given above.

(Hint: Instead of using induction, proceed by cases, and refer to Theorem 6.1.2.)

Show $r_a = r_b$:

$4 \mid a-b$

$\Rightarrow 4 \mid q_a \cdot 4 + r_a - q_b \cdot 4 + r_b$

$\Rightarrow 4 \mid 4(q_a - q_b) + r_a - r_b$

since $4 \mid 4(q_a - q_b)$, it follows that

$4 \mid r_a - r_b$.

$0 \le r_a \le 3$, $0 \le r_b \le 3$

so $-3 \le r_a - r_b \le 3$

the only integer between -3 and 3 that is divisible by 4 is 0.

So $r_a - r_b = 0 \Rightarrow r_a = r_b$

2) Define, on the set of integers, the following equivalence relation.

$k \sim l$ if and only if $|k| = |l|$.

(a) Prove that the above relation is indeed an equivalence relation.

- Reflexive: $|k| = |k| \quad \forall k \in \mathbb{Z}$.
- Symmetric: if $|k| = |l|$, then $|l| = |k|$. $\forall k, l \in \mathbb{Z}$.
- Transitive: if $|k| = |l|$, and $|l| = |m|$, then $|k| = |m| \quad \forall k, l, m \in \mathbb{Z}$.

(b) Describe the equivalence classes for this relation.

Definition 7.3.1. Let R be an equivalence relation on a set S, and $x \in S$.
The **equivalence class of** x is the set of all elements $y \in S$, which are equivalent to x:

$$\{y \in S : (x, y) \in R\}.$$

We denote the equivalence class of x by $[x]$.

$\ldots, -4, -3, -2, -1, \boxed{0}, 1, 2, 3, 4, \ldots$

$[0] = \{0\}$.

$[1] = \{-1, 1\} = [-1]$

$[2] = \{-2, 2\} = [-2]$

3) Let $f : A \to B$ be an arbitrary function. Prove that the relation

$x \sim y$ if and only if $f(x) = f(y)$,

on the set A, is an equivalence relation.

Reflexive: $f(x) = f(x) \quad \forall x \in A$.

Symmetric: if $f(x) = f(y)$, then $f(y) = f(x) \quad \forall x, y \in A$.

Transitive: if $f(x) = f(y)$, and $f(y) = f(z)$, then $f(x) = f(z) \quad \forall x, y, z \in A$.