w5

08 June 2021 11:53

topics:
- proof methods...? (direct, contrapositive, contradiction)
- more logic symbols!
- proving some polynomial has no rational solutions

1) **There are infinitely many prime numbers.** This is Theorem 3.6.6 of the course note.

a) Make sure the proof of it is clear. What type of proof is used? Contradiction

b) Is it possible to use other two types of proof to prove it. Discuss.

a) Proof by contradiction: suppose there are finitely many prime numbers $\{p_1, p_2, \cdots, p_n\}$. Consider $x = (p_1 \cdot p_2 \cdot \cdots \cdot p_n) + 1$

$\{2, 3, 5\}$

x is $> p_1, p_2, \cdots, p_n$ so x is composite. So there is some prime p_i that divides x.

$x = 2 \cdot 3 \cdot 5 + 1 = 31$

Since p_i divides x, and p_i divides $(p_1 \cdot p_2 \cdots p_n)$ (since it's in the product)

so p_i divides $x - (p_1 \cdot p_2 \cdots p_n) = 1$, a contradiction.

b) No! Direct proof and contrapositive proof only allow us to prove statements "$P \Rightarrow Q$"

2) (a) Write the following statement using mathematical symbols.

For $\underbrace{\text{any}}_{\forall}$ positive real number x, $\underbrace{\text{there is}}_{\exists}$ natural number n, for which $\frac{1}{n} < x$.

$(\forall x \in \mathbb{R}, x > 0)(\exists n \in \mathbb{N})(\frac{1}{n} < x)$. $\qquad (\forall x \in \mathbb{R})[x > 0 \Rightarrow (\exists n \in \mathbb{N})(\frac{1}{n} < x)]$

(b) Using mathematical symbols write the negation of the above statement. Your answer should not include $\neg$.

$(\exists x \in \mathbb{R}, x > 0)(\forall n \in \mathbb{N})(\frac{1}{n} \geq x)$ $\qquad (\exists x \in \mathbb{R})[x > 0 \wedge (\forall n \in \mathbb{N})(\frac{1}{n} \geq x)]$

(c) Which one of the statements in parts (a) or (b) is true?

(a) is true. "Archimedian Principle"

Proof of $(\forall x \in \mathbb{R}, x > 0)(\exists n \in \mathbb{N})(\frac{1}{n} < x)$.

Let $x \in \mathbb{R}, x > 0$. WTF $n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$.

Take $n = \lceil \frac{1}{x} \rceil + 1$ RW: $\frac{1}{n} < x$ $\quad \frac{1}{nx} < 1$ $\quad \frac{1}{x} < n$

$\frac{1}{\lceil \frac{1}{x} \rceil + 1} < \frac{1}{\lceil \frac{1}{x} \rceil} \leq \frac{1}{\frac{1}{x}} = x$. $\qquad \lceil \frac{1}{x} \rceil + 1$ works.

$\lceil 1 \rceil = 1$

$\lceil \frac{3}{2} \rceil = 2$

3) Prove that the equation $x^3 + x + 1 = 0$ have no rational solutions.

$x = \frac{p}{q}$

Hint: Contradiction. Suppose there is some rational (p/q) such that (p/q)^3 + (p/q) + 1 = 0...

Assume $\frac{p}{q}$ is in lowest terms (since every rational has a lowest-terms fraction representation)

$(\frac{p}{q})^3 + \frac{p}{q} + 1 = 0$. $\quad p, q \in \mathbb{Z}$.

representation)

$\left(\frac{p}{q}\right)^3 + \frac{p}{q} + 1 = 0$. $p, q \in \mathbb{Z}$.

$\frac{p^3}{q^3} + \frac{p}{q} + 1 = 0$

$\underbrace{p^3 + pq^2 + q^3}_{\text{integer}} = 0$

Since $p, q \in \mathbb{Z}$, we have four cases:

Case 1: p, q both odd.

p^3 is odd, pq^2 is odd, q^3 is odd

$p^3 + pq^2 + q^3$ is the sum of 3 odd numbers, so odd

$p^3 + pq^2 + q^3 = 0$ which is even.

So p and q can't both be odd.

Case 2: p odd, q even.

p^3 odd, pq^2 even, q^3 even

$p^3 + pq^2 + q^3$ odd, but 0 even.

Case 3: p even, q odd

p^3 even, pq^2 even, q^3 odd.

$p^3 + pq^2 + q^3$ odd, but 0 even.

Case 4: p, q both even.

2 factors into both p and q.

So $\frac{p}{q}$ isn't in lowest terms,

contradicting our assumption.

3.7.31. Prove that the following equations have **no rational** solutions.

(a) $x^3 + x^2 = 1$

(b) $x^3 + x + 1 = 0$ (same steps)

(c) $x^5 + 3x^3 + 7 = 0$

(d) $x^5 + x^4 + x^3 + x^2 + 1 = 0$