w6

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hey! hope you had a nice break :)
i... just played video games during the break. don't ask me.

hmm... so the lecture has covered induction, and this is the focus of today's tutorial.
topics:
- proving a sum identity using simple induction
- proving divisibility of a sequence of numbers using induction (over the even numbers!)
- sum and product notation

1) Prove for all $n \in \mathbb{N}$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

$P(n)$: "$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$"

WTS $P(n)$ $\forall n \in \mathbb{N}$.

Base case $(n=1)$: $1^2 = \frac{4(1)^3-1}{3}$ ✓

Induction step: Assume $P(n)$. WTS $P(n+1)$.

$1^2 + 3^2 + \cdots + (2n-1)^2 + (2n+1)^2$

$= (1^2 + 3^2 + \cdots + (2n-1)^2) + (2n+1)^2$

IH $= \frac{4n^3-n}{3} + (2n+1)^2$

$= \frac{4n^3-n}{3} + 4n^2 + 4n + 1 = \frac{4n^3 + 12n^2 + 11n + 3}{3} = \frac{4(n^3+3n^2+3n+1) - (n+1)}{3} = \frac{4(n+1)^3 - (n+1)}{3}$

So $P(n+1)$ is true.

$P(1)$: "$1^2 = \frac{4(1)^3-1}{3}$"

Assume $\rightarrow P(2)$: "$1^2 + 3^2 = \frac{4(2)^3 - 2}{3}$"

Want to prove $\rightarrow P(3)$: "$1^2 + 3^2 + 5^2 = \frac{4(3)^3 - 3}{3}$"

$\vdots$

2) Prove, by induction, that $10^n - 1$ is divisible by 11 for every even natural number n.

$P(n)$: "$10^n - 1$ is divisible by 11"

Want to show $P(n)$ $\forall n \in \mathbb{N}$, n even.

Base case $(n=2)$: $10^2 - 1$ is divisible by 11 ✓

IS: Assume $P(n)$. WTS $P(n+2)$.

$10^{n+2} - 1$

$= 10^2 \cdot 10^n - 1$

$= 100 \cdot 10^n - 1$

$= (99+1) \cdot 10^n - 1$

$= \underbrace{99 \cdot 10^n}_{\text{divisible by 11}} + \underbrace{10^n - 1}_{\text{divisible by 11 (IH)}}$ so $10^{n+2} - 1$ is divisible by 11.

So $P(n+2)$ holds.

3) Compute the following expressions (obtain a single number).

hint: write it out!

a) $\sum_{n=1}^{100}(n.(-1)^n) = 1\cdot -1 + 2\cdot 1 + 3\cdot -1 + 4\cdot 1 + \dots + 99\cdot -1 + 100\cdot 1$

$= -1 + 2 + -3 + 4 + \dots + -99 + 100$

$= (-1+2) + (-3+4) + \dots + (-99+100) = \underbrace{1+1+\dots+1}_{50} = 50.$

b) $\prod_{k=1}^{69} 2^{k-35} =$

$2^{1-35}\cdot 2^{2-35}\cdot \dots \cdot 2^{34-35}\cdot 2^{35-35}\cdot 2^{36-35}\cdot \dots \cdot 2^{68-35}\cdot 2^{69-35}$

$= \cancel{2^{-34}}\cdot \cancel{2^{-33}}\cdot \dots \cdot \cancel{2^{-1}}\cdot 2^{0}\cdot \cancel{2^{1}}\cdot \dots \cdot \cancel{2^{33}}\cdot \cancel{2^{34}} = 1$

c) $\prod_{k=10}^{99}\frac{k}{k+1} = \frac{10}{\cancel{11}}\cdot\frac{\cancel{11}}{\cancel{12}}\cdot\frac{\cancel{12}}{\cancel{13}}\cdot \dots \cdot \frac{\cancel{98}}{\cancel{99}}\cdot\frac{\cancel{99}}{100} = \frac{10}{100} = \frac{1}{10}.$