w7

13 July 2021 11:23

hmm... today's tutorial will be kinda short. :p

topics:
- strong induction

yea that's it. ask any questions you have!

1) Check the proof of Theorem 4.5.4 again. The theorem states: every natural number n can be expressed as a sum of distinct nonnegative integer powers of 2.
Explain why it is essential to apply strong induction.

try a few examples! how would the induction step work for k = 8, and k = 9, for example?

Theorem 4.5.4. *Every natural number n can be expressed as a sum of distinct nonnegative integer powers of 2.*[3]

Proof. The base case is easily verified, as $1 = 2^0$.

Let $k \in \mathbb{N}$, and assume that the theorem holds true for $n = 1, 2, 3, \ldots, k$. We need to prove that $k+1$ can be expressed as a sum of distinct nonnegative integer powers of 2, and we do that by looking at the following two possible cases.

- **Case 1: k is even.**

  By assumption, the theorem applies to $n = k$, and so we can write

  $$k = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m},$$

  where $a_1, \ldots, a_m$ are distinct **positive** integers. As k is even, the term 2^0 does not appear in the sum. Consequently,

  $$k + 1 = 2^0 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m},$$

  and we have expressed $k+1$ in the required form.

- **Case 2: k is odd.**

  The argument used in Case 1 won't work here (why?), so we use a different approach. As k is odd, $k+1$ is even, and we can write $k+1 = 2m$, for some $m \in \mathbb{N}$. As m is smaller than $k+1$, the induction hypothesis applies, and we have

  $$m = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_m},$$

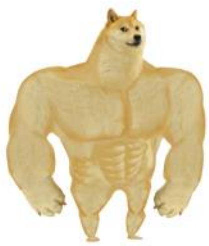
  for some nonnegative distinct integers $a_1, \ldots, a_m$ (this time, one of the a_i's may be zero!).

  We conclude that

  $$k + 1 = 2m = 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_m+1},$$

  as needed. Note that since $a_1, a_2, \ldots, a_m$ are distinct, so are $a_1 + 1, a_2 + 1, \ldots, a_m + 1$.

We proved the theorem for $n = k + 1$, and hence, by strong induction, for any $n \in \mathbb{N}$. □



Assume P(1), P(2), ..., P(k)

assum P(k)

$P(1)$: $1 = 2^0$

$P(2)$ $2 = 2^1$

$\vdots$

$P(7)$ $7 = 2^2 + 2^1 + 2^0$

$P(8)$ $8 = 2^3$

$k = 7$

in case 2, we don't actually assume that m = k. all we know is that 1 <= m <= k. if we use simple induction instead, then the induction hypothesis (P(k)) might not necessarily apply to m, as m might not be equal to k; while in strong induction as long as 1 <= m <= k (and m is an integer) we may apply the induction hypothesis.

2) Check the proof for Theorem 4.5.1 again. The theorem states: every natural number $n \geq 2$ can be written as a product of prime numbers.
Explain how the induction hypothesis is applied to prove the claim.

Theorem 4.5.1. *Every natural number $n \geq 2$ can be written as a product of prime numbers.*

again, try some examples! how would the induction step apply to k = 7 and k = 18, for example?

Proof. We use strong induction in our proof, and $n = 2$ as our base case.

If $n = 2$, the theorem is valid, as 2 is a prime number. Assume that the theorem holds true for $n = 2, 3, 4, \ldots, k$ (for some natural number $k \geq 2$), and consider $n = k + 1$. If $k + 1$ happens to be a prime number, the theorem applies. Otherwise, $k + 1$ is composite, and is divisible by some natural number $2 \leq m \leq k$. Equivalently, $k + 1 = m \cdot \ell$, where m, ℓ are natural numbers between 2 and k.

Both m and ℓ are natural numbers, greater than 1 and smaller than $k + 1$, and hence covered by the induction hypothesis. Therefore, m and ℓ are products of prime numbers, and consequently, so is $k + 1 = m \cdot \ell$.

By PSMI, the theorem is valid for all natural numbers $n \geq 2$. □

$k = 7$

$P(2), \ldots, P(7)$

WTP $P(8)$

4 divides 8

$8 = 4 \cdot 2 = 2 \cdot 2 \cdot 2$

because m and l need to be covered by the induction hypothesis! we don't know if m = k or l = k; all we know is that 2 <= m <= k

$k + 1 = m \cdot l$.

By PSMI, the theorem is valid for all natural numbers $n \geq 2$. □

because m and l need to be covered by the induction hypothesis! we don't know if m = k or l = k; all we know is that 2 <= m <= k.

$8 = 4 \cdot 2 = 2 \cdot 2 \cdot 2$

IH IH

$4 = 2 \cdot 2$

3) Let (a_n) be a sequence satisfying $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Given that a_1 and a_2 are odd, prove that a_n is odd for $n \in \mathbb{N}$.

Want to show $P(n)$ $\forall n \in \mathbb{N}$

where $P(n)$: "a_n is odd"

Base case: $P(1)$ a_1 is odd ✓

$P(2)$ a_2 is odd ✓

as a general rule of thumb, the number of base cases you need is equal to the number of hypotheses you need for your induction step.
(in our case, we need P(k - 1) and P(k) in our induction step, so we need 2 base cases)

Induction step: Assume ~~$P(1), \ldots, P(k)$~~

$P(k-1), P(k)$.

a_{k-1} odd a_k odd

WTS $P(k+1)$: "a_{k+1} is odd" $k \geq 2$

since $k \geq 2$, $k+1 \geq 3$, so $a_{k+1} = \underbrace{2a_k}_{\text{even}} + \underbrace{3a_{k-1}}_{\text{odd}}$ = something odd.

(since a_{k-1} odd)

This shows $P(k+1)$.