w8

19 July 2021 14:32

bijections! n stuff.
(they also want you to use some calculus...)
here's a croissant. i am hungry.

$f: A \to B$ bijection $\Leftrightarrow$ f injection and surjection

$f: A \to B$ injection $\Leftrightarrow$ $(\forall x, y \in A)\ f(x) = f(y) \Rightarrow x = y$

$f: A \to B$ surjection $\Leftrightarrow$ $(\forall y \in B)(\exists x \in A)\ f(x) = y$.

1) Which one of the following functions is a bijection? Explain.

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^6$

not a bijection: it's neither inj. nor surj.

f not inj: $-1 \neq 1$
but $f(-1) = f(1)$
cuz $(-1)^6 = 1^6 = 1$

f not surj: if $y = -1$ (in the codomain)
there is no $x \in$ domain
s.t. $f(x) = y$. (since $x^6 \geq 0$, so $x^6 \neq -1$)

$g: [0, \infty) \to [0, \infty)$
$g(x) = x^6$

it's bijective.

g is injective: if $g(x) = g(y)$ (where $x, y \in [0, \infty)$)
then $x^6 = y^6 \Rightarrow x = \pm y$. and $x \neq -y$ since $x, y \geq 0$.
(unless $x = y = 0$). So $x = y$.

g surjective: Let $y \in [0, \infty)$.
Let $x = \sqrt[6]{y} \in [0, \infty)$
Then $g(x) = x^6 = y$.

$h: \mathbb{Z} \to \mathbb{Z}$
$h(x) = x^6$ not bijective!

h is not inj: $h(-1) = h(1) = 1$
h is not surj: Let $y = -1$. $h(x) \geq 0$ so $h(x) \neq -1$ for all $x \in \mathbb{Z}$.

$p: \mathbb{Q} \to [0, \infty)$
$p(x) = x^6$ not bijective!

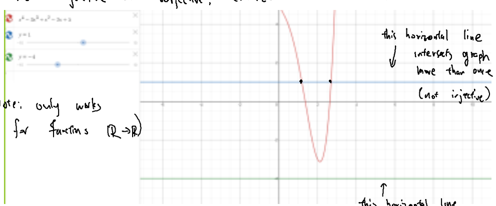
p not inj.: $p(-1) = p(1) = 1$
p not surj: Let $y = 2$. if $p(x) = 2$, it must be the case that
$x = \pm \sqrt[6]{2}$ but $\pm \sqrt[6]{2} \notin \mathbb{Q}$

2) Using desmos.com investigate if the following polynomials are injective or surjective. Please note while desmos (or any other graphing tool) can be helpful to make a good observation it is NOT a way to prove any mathematical claims. No proof based on the graph will be accepted in this course. This exercise is for observation only.

a) $p(x) = x^4 - 3x^3 + x^2 - 2x + 5$
b) $q(x) = x^5 + x^2 - 2x + 3$

$p: \mathbb{R} \to \mathbb{R}$ not injective or surjective. (horizontal line test)
$q: \mathbb{R} \to \mathbb{R}$

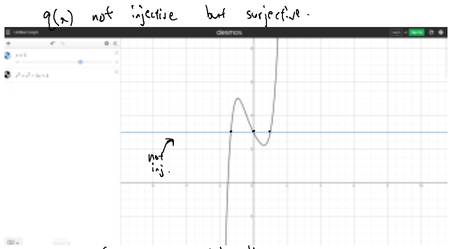
(Note: only works for functions $\mathbb{R} \to \mathbb{R}$)



this horizontal line intersects graph more than once (not injective)

this horizontal line doesn't intersect graph (not surj.)

$q(x)$ not injective but surjective.



not inj.

(every horizontal line intersects graph, so surjective)

3) Can we say any polynomial of an odd degree from $\mathbb{R}$ to $\mathbb{R}$ is surjective? Can we say any polynomial of an even degree from $\mathbb{R}$ to $[0, \infty)$ is surjective?

You might need some calculus and the Intermediate Value Theorem, if you are not familiar with it you can skip it.

Every polynomial of odd degree is surjective.* Intuitively, polynomials of odd degree "span" the whole y axis; every horizontal line has to intersect the graph somewhere.

Polynomials of even degree from R to [0, infinity) need not be surjective. Take p(x) = x^2 + 1 for example: this never reaches 0 (which is in the codomain), so this isn't surjective.

Formal proof of * using the intermediate value theorem (IVT):

Suppose $p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots + a_1 x + a_0$ is a polynomial of odd degree
(k is odd, $a_k \neq 0$).

Let $y \in \mathbb{R}$. We find $x \in \mathbb{R}$ s.t. $p(x) = y$.

Case 1: $a_k > 0$.

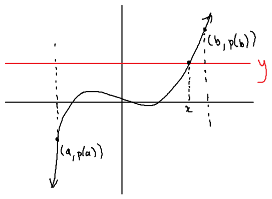
Then $\lim_{x \to \infty} p(x) = \infty$. So there exists $b \in \mathbb{R}$ s.t. $p(b) \geq y$.
we may choose $b > 0$ if necessary.

$\lim_{x \to -\infty} p(x) = -\infty$. So there exists $a \in \mathbb{R}$ s.t. $p(a) \leq y$.
we may choose $a < 0$ if necessary.

(this gives us $a < b$)

Since $a<b$ and $p(a) \le y \le p(b)$, and p is continuous,
the IVT applies, so there is some $z \in [a,b]$ s.t.
$p(z) = y$. This shows p is surjective.

Picture:
(to aid proof)



Case 2: $a_k < 0$.
do the same thing!
but flipped.

4) Consider the following functions: $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(m) = (2m, m-1)$ and $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $g(m,n) = |m.n|$.

(a) Is f injective? Explain.

$g \circ f(x) = g(f(x))$

(b) Is g surjective? Explain.

"circ"

(c) State the domain and the codomain of $g \circ f$, and write a formula for this composition.

(d) State the domain and the codomain of $f \circ g$, and write a formula for this composition.

(a) f is injective: Suppose $m, n \in \mathbb{Z}$ $f(m) = f(n)$.
$\Rightarrow (2m, m-1) = (2n, n-1)$
$\Rightarrow$ $2m = 2n$ and $m-1 = n-1$
$\Rightarrow$ $m = n$.

(b) g is NOT surjective: Let $y = -1$.

$g(m,n) = |m \cdot n| \geq 0 \neq -1$
for any $(m,n) \in \mathbb{Z} \times \mathbb{Z}$.

(c) $g \circ f : \underbrace{\mathbb{Z}}_{\text{domain of } f} \to \mathbb{Z}$ ← codomain of g

$(g \circ f)(m) = g(f(m)) = g(2m, m-1) = |2m \cdot (m-1)| = |2m^2 - 2m|$

(d) $f \circ g : \underbrace{\mathbb{Z} \times \mathbb{Z}}_{\text{domain of } g} \to \underbrace{\mathbb{Z} \times \mathbb{Z}}_{\text{codomain of } f}$, $(f \circ g)(m,n) = f(g(m,n)) = f(|m \cdot n|) = (2|m \cdot n|, |m \cdot n| - 1)$.