w9

26 July 2021 18:04

hello again!

topics:

- cardinality (finite, countable, uncountable sets)

note: in this course, "countable" always means countably infinite (having the same cardinality as the natural numbers). in other courses you may see "countable" refer to both finite and countably infinite sets.

1) For each of the following sets, decide wether it is finite, countable, or uncountable. Explain your answer.

Theorem 5.4.5 (Cantor's Theorem). *Let X be any set. Then the sets X and $P(X)$* **do not** *have the same cardinality.*

$P(\mathbb{N})$ uncountable. $(|P(\mathbb{N})| > |\mathbb{N}|)$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \cap [0.03, 1] = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{33}\}$ 0.0294117647059

finite.

$\mathbb{Q}$ countable

Theorem 5.4.1. *The set of natural numbers, $\mathbb{N}$, and the set of rational numbers, $\mathbb{Q}$, have the same cardinality.*

$(0, \infty)$

uncountable. $|(0,\infty)| = |\mathbb{R}|$ (eg. e^x is a bijection $\mathbb{R} \to (0,\infty)$)

$P(\mathbb{Z})$

uncountable $|P(\mathbb{Z})| > |\mathbb{Z}| = |\mathbb{N}|$

$(2, 3)$

uncountable $|(2,3)| = |(0,\infty)|$

All prime numbers

countable

$$\begin{array}{cccccccc} \mathbb{N} & 1 & 2 & 3 & 4 & 5 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{primes} & 2 & 3 & 5 & 7 & 11 & \cdots \end{array}$$

(any infinite subset of $\mathbb{N}$ is countable)

$\mathbb{N} \cap (-\infty, 1000)$

$= \{1, 2, \cdots, 1000\}$ finite.

2) Fill the blanks (—) with $\in$ or $\subseteq$. The set $A = \{1, 2, \{1, 2\}\}$.

$\emptyset \subseteq \mathbb{Z}$

$\{\{1\}\} \subseteq P(A)$

$\{1, \{1, 2\}\} \in P(A)$

$\{1, 2\} \in P(A)$

$\{\{1, 2\}\}$ both $P(A)$

$\mathbb{N} \in P(\mathbb{Z})$

$P(\mathbb{N}) \subseteq P(\mathbb{Z})$

$\{1, \{1,2\}\} \in P(A) \iff \{1, \{1,2\}\} \subseteq A$.

$P(A) = \{S : S \subseteq A\}$

$= \{\emptyset, \{1\}, \{2\}, \{\{1,2\}\}, \{1,2\}, \{1,\{1,2\}\}, \{2,\{1,2\}\}, \{1,2,\{1,2\}\}\}$

$\mathbb{N} = \{1,2,3,\ldots\}$ $P(\mathbb{Z}) = \{$ all subsets of $\mathbb{Z}\}$ $\mathbb{N} \subseteq \mathbb{Z} \iff \mathbb{N} \in P(\mathbb{Z})$.

3) Let $f : \mathbb{N} \to P(\mathbb{N})$ be given by $f(n) = \{n + 1, n + 2, n + 3, \cdots\}$.

f takes in a natural number and outputs a subset of the natural numbers

(a) Find the set $f(3) \cap [-8, 8]$.

$f(3) = \{4, 5, 6, \ldots\}$ $f(3) \cap [-8, 8] = \{4, 5, 6, 7, 8\}$.

(b) Is f an injection? Explain.

(c) Is f a surjection? Explain.

(b) yes.

Proof of "if $f(n) = f(m)$ $(n, m \in \mathbb{N})$, then $n = m$"

Contrapositive: if $n \neq m$, then $f(n) \neq f(m)$.

Assume $n \neq m$.

case 1: $n < m$, so $n+1 < m+1$

$f(n) = \{n+1, n+2, \ldots, m+1, \ldots\}$ $f(m) = \{m+1, \ldots\}$

$n+1 \in f(n)$ $n+1 \notin f(m)$

So $f(n) \neq f(m)$ (they can't be the same set).

Case 2: $m < n$

same thing.

(c) no. $\phi \in P(\mathbb{N})$

but there does not exist $n \in \mathbb{N}$ s.t. $f(n) = \phi$.

Since $n+1 \in f(n)$ always.