Problem 1

Let $p(x) = b^2x^2 - (b-1)x - \frac{1}{4}$. Determine, for which $b \in \mathbb{R}$,[a] does $p(x)$:

(a) have no roots.

(b) have exactly one root.

(c) have two roots.

[a] $\mathbb{R}$ stands for the **real numbers**.

Problem 2

Consider the following "proof" of the mathematical statement "for all $x \neq 0$, $\frac{49}{x^2} + 5 + x^2 \geq 21$".

Proposition. For all $x \neq 0$, $\frac{49}{x^2} + 5 + x^2 \geq 21$.

Proof.

$$\begin{aligned}
&\frac{49}{x^2} + 5 + x^2 \geq 21 \\
\Rightarrow &\frac{49}{x^2} + x^2 \geq 16 \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq 8 \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq 7 \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq \sqrt{49} \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq \sqrt{\frac{49}{x^2} \cdot x^2} \qquad \text{(arithmetic-geometric mean inequality).}
\end{aligned}$$

Identify some mathematical proof quality issues with the above "proof". Is the original statement true? If so, rewrite the proof of the statement; if not, modify the statement so that it is true, and then rewrite the proof of the statement.

Problem 3

If $a, b > -1$, prove that $\frac{a+b}{2} + 1 \geq \sqrt{(a+1)(b+1)}$.