Problem 1

Let $p(x) = b^2x^2 - (b-1)x - \frac{1}{4}$. Determine, for which $b \in \mathbb{R}$,[a] does $p(x)$:

(a) have no roots.

(b) have exactly one root.

(c) have two roots.

Solution To apply the quadratic formula to find the number of roots of $ax^2 + bx + c$, we need to first ensure that $a \neq 0$. So we have two cases:

- $b = 0$: then $p(x) = x - \frac{1}{4}$, so $p(x)$ has one root (since it's a linear function with nonzero slope).
- $b \neq 0$: then $b^2 \neq 0$ so we may apply the quadratic formula. The discriminant of $p(x)$ is

$$\Delta = (-(b-1))^2 - 4b^2\left(-\frac{1}{4}\right) = (b-1)^2 + b^2.$$

  We notice that the discriminant is always nonnegative, since $(b-1)^2, b^2 \geq 0$. Even more precisely, the discriminant is always positive, since for $(b-1)^2 + b^2 = 0$ to hold, both $(b-1)^2 = 0$ and $b^2 = 0$ must hold simultaneously which is impossible.

  Since the discriminant is always positive, $p(x)$ has two roots.

[a] $\mathbb{R}$ stands for the **real numbers**.

Problem 2

Consider the following "proof" of the mathematical statement "for all $x \neq 0$, $\frac{49}{x^2} + 5 + x^2 \geq 21$".

Proposition. For all $x \neq 0$, $\frac{49}{x^2} + 5 + x^2 \geq 21$.

Proof.

$$\begin{aligned}
&\frac{49}{x^2} + 5 + x^2 \geq 21 \\
\Rightarrow &\frac{49}{x^2} + x^2 \geq 16 \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq 8 \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq 7 \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq \sqrt{49} \\
\Rightarrow &\frac{\frac{49}{x^2} + x^2}{2} \geq \sqrt{\frac{49}{x^2} \cdot x^2} \qquad \text{(arithmetic-geometric mean inequality).}
\end{aligned}$$

Identify some mathematical proof quality issues with the above "proof". Is the original statement true? If so, rewrite the proof of the statement; if not, modify the statement so that it is true, and then rewrite the proof of the statement.

Solution Here are some issues with the proof:

- **The proof starts with the statement to be proven, and derives a true statement from the statement to be proven.** Instead, a mathematical proof should always start with a statement already known to be true (such as the AGM inequality), and use this statement to derive the statement to be proven.
- Language is rarely used to guide the reader; the proof mostly consists of a list of inequalities.
- The derivation of the inequalities is somewhat unclear.[a]

Indeed, the statement that was "proven" is not true. We can see this by substituting $x = \sqrt{7}$:[b]

$$\frac{49}{\sqrt{7}^2} + 5 + \sqrt{7}^2 = 7 + 5 + 7 = 19 \not\geq 21.$$

However, we can prove the following statement: "for all $x \neq 0$, $\frac{49}{x^2} + 5 + x^2 \geq 19$". We will first perform some rough work in a style similar to the incorrect proof, but this rough work will be excluded from our actual proof.

Rough Work

$$\frac{49}{x^2} + 5 + x^2 \geq 19$$
$$\Rightarrow \frac{49}{x^2} + x^2 \geq 14$$

An alternative way to write the AGM inequality is $a + b \geq 2\sqrt{ab}$ (assuming $a, b \geq 0$).[c] Indeed, applying this form of the AGM inequality to the above (which is allowed since $\frac{49}{x^2}, x^2 \geq 0$),

$$\frac{49}{x^2} + x^2 \geq 2\sqrt{\frac{49}{x^2} \cdot x^2} = 2\sqrt{49} = 14.$$

Now we can write the proof.

Proof. Since $\frac{49}{x^2}, x^2 \geq 0$, we may use the AGM inequality to obtain

$$\frac{\frac{49}{x^2} + x^2}{2} \geq \sqrt{\frac{49}{x^2} \cdot x^2} = 7.$$

Multiplying both sides by 2, then adding 5 to both sides,

$$\frac{49}{x^2} + 5 + x^2 \geq 2 \cdot 7 + 5 = 19.$$

This completes the proof. ■

[a] What constitutes a "clear" derivation will depend on audience and context; you will have to balance clarity and conciseness depending on your target audience.

[b] I got this value for x by attempting to minimize $\frac{49}{x^2} + 5 + x^2$ so that it's not ≥ 21. Minimizing $\frac{49}{x^2} + 5 + x^2$ is the same thing as minimizing $\frac{1}{2}\left(\frac{49}{x^2} + x^2\right)$ (why?), and we know from the AGM inequality that $\frac{1}{2}\left(\frac{49}{x^2} + x^2\right)$ is minimized when $\frac{49}{x^2} = x^2$, or $x^4 = 49$ which gives $x = \pm\sqrt{7}$.

[c] This may be useful for spotting where the AGM inequality can be used.

Problem 3

If $a, b > -1$, prove that $\frac{a+b}{2} + 1 \geq \sqrt{(a+1)(b+1)}$.

Solution

Rough Work

$$\frac{a+b}{2} + 1 \geq \sqrt{(a+1)(b+1)}$$
$$\Rightarrow \frac{a+b}{2} + \frac{2}{2} \geq \sqrt{(a+1)(b+1)}$$
$$\Rightarrow \frac{a+1+b+1}{2} \geq \sqrt{(a+1)(b+1)}$$

The above is just the AGM inequality applied to $a+1$ and $b+1$. The square root version of the AGM inequality can be applied since $a, b > -1$ ensures $a+1, b+1 \geq 0$.

Proof. Since $a, b > -1$, we have $a+1, b+1 > 0$, so by the AGM inequality,

$$\frac{(a+1)+(b+1)}{2} \geq \sqrt{(a+1)(b+1)}.$$

We may rewrite the left side to obtain

$$\frac{a+b}{2} + \frac{2}{2} \geq \sqrt{(a+1)(b+1)}$$

or

$$\frac{a+b}{2} + 1 \geq \sqrt{(a+1)(b+1)}$$

as needed. ■

Comment. An alternative proof involves square both sides and expanding, but this approach is more tedious.