Problem 1
Let A, B be subsets of some universal set U. Prove that $A \setminus B^C = A \cap B$.
Solution
Proof. We must show that $A \setminus B^C \subseteq A \cap B$ and $A \cap B \subseteq A \setminus B^C$.

- $A \setminus B^C \subseteq A \cap B$: Suppose $x \in A \setminus B^C$. Then $x \in A$ and $x \notin B^C$. Since $x \notin B^C$, we have $x \in (B^C)^C$. But $B = (B^C)^C$, so $x \in B$. Combining $x \in A$ with $x \in B$, we have $x \in A \cap B$ as needed.
- $A \cap B \subseteq A \setminus B^C$: Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $B = (B^C)^C$, $x \in (B^C)^C$ as well, so $x \notin B^C$. Combining $x \in A$ with $x \notin B^C$, we have $x \in A \setminus B^C$.

Comment. You may notice that our proof of $A \cap B \subseteq A \setminus B^C$ is just our proof of $A \setminus B^C \subseteq A \cap B$ written backwards. A lot of set identity proofs are like this; however in general when we are proving two sets are equal, this strategy of writing backwards may not work (as some parts of your proof may no longer be true when written backwards).

Problem 2

1. Let $S = \{1, 2, 3\}$. List all subsets of S.
2. Let S be a finite set with n elements. How many distinct subsets of S are there?

Solution

1. There are 8 subsets in total: $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.
2. There are 2^n distinct subsets of S. Labelling the elements of S as $s_1, s_2, \ldots, s_n$, notice that to create a subset of S, we are given two choices on whether to include s_1 in our subset, another two choices on whether to include s_2 in our subset, and so on. In total, we are given 2 choices for each of the n elements of S; thus there are 2^n total subsets we can create.

Problem 3
Let $f : (2, \infty) \to \mathbb{R}, f(x) = \dfrac{x+1}{x^2 - 4}$. Prove that the range of f, $f((2, \infty))$, is $(0, \infty)$.
Solution
Proof.

- $f((2, \infty)) \subseteq (0, \infty)$: Let $y \in f((2, \infty))$. Then by definition there is some $x \in (2, \infty)$ such that $f(x) = y$. Rewriting,
$$y = \frac{x+1}{x^2 - 4}.$$
Notice that $x + 1 > 0$ since $x > 2$, and $x^2 - 4 > 0$ since $x > 2$. Thus y is a ratio of positive numbers, which shows $y > 0$.
- $(0, \infty) \subseteq f((2, \infty))$: Let $y \in (0, \infty)$. We want to find an $x \in (2, \infty)$ such that $f(x) = y$.

Rough Work

$$\begin{aligned} y &= \frac{x+1}{x^2 - 4} \\ \Rightarrow & yx^2 - 4y = x + 1 \\ \Rightarrow & yx^2 - x - 4y - 1 = 0 \end{aligned}$$

So to find $x \in (2, \infty)$ such that $f(x) = y$, we could attempt to solve for x in the above quadratic. The discriminant is $(-1)^2 - 4y(-4y-1) = 16y^2 + 4y + 1$ which is always > 0 since $y \in (0, \infty)$, so there are two solutions to the quadratic:

$$x = \frac{1 \pm \sqrt{16y^2 + 4y + 1}}{2y}.$$

Recall that we want a solution $x \in (2, \infty)$; to make it more likely for $x > 2$, we should always take the greater of the two solutions. Thus let us define

$$x = \frac{1 + \sqrt{16y^2 + 4y + 1}}{2y}.$$

It remains to verify that $x > 2$ (for any value of y), and $f(x) = y$. Let us perform some rough work to show that $x > 2$:

$$\begin{aligned}
&x > 2 \\
\Rightarrow &\frac{1 + \sqrt{16y^2 + 4y + 1}}{2y} > 2 \\
\Rightarrow &1 + \sqrt{16y^2 + 4y + 1} > 4y && \text{(multiplying both sides by } 2y \text{ which is } > 0) \\
\Rightarrow &\sqrt{16y^2 + 4y + 1} > 4y - 1
\end{aligned}$$

Now if $4y - 1$ is negative (when $y \in [0, \frac{1}{4})$), the above inequality holds (since the left side is always positive). Otherwise $4y - 1 \geq 0$ so we can square both sides to get

$$16y^2 + 4y + 1 > 16y^2 - 8y + 1$$

which is true since $y > 0$.

Define

$$x = \frac{1 + \sqrt{16y^2 + 4y + 1}}{2y}.$$

First of all, we need to show that $x \in (2, \infty)$. Since $y > 0$, we have

$$\begin{aligned}
&16y^2 + 4y + 1 > 16y^2 - 8y + 1 \\
\Rightarrow &16y^2 + 4y + 1 > (4y - 1)^2 \\
\Rightarrow &\sqrt{16y^2 + 4y + 1} > |4y - 1| \\
\Rightarrow &\sqrt{16y^2 + 4y + 1} > 4y - 1 && \text{since } |4y - 1| > 4y - 1 \\
\Rightarrow &1 + \sqrt{16y^2 + 4y + 1} > 4y \\
\Rightarrow &\frac{1 + \sqrt{16y^2 + 4y + 1}}{2y} > 2 && \text{since } y > 0 \text{ we can divide both sides by } 2y \\
\Rightarrow &x > 2.
\end{aligned}$$

It remains to verify that $f(x) = y$. Indeed,

$$\begin{aligned} f(x) &= \frac{x+1}{x^2-4} \\ &= \frac{\frac{1+\sqrt{16y^2+4y+1}}{2y}+1}{\left(\frac{1+\sqrt{16y^2+4y+1}}{2y}\right)^2-4} \\ &= \frac{\frac{1+\sqrt{16y^2+4y+1}}{2y}+1}{\frac{16y^2+4y+2+2\sqrt{16y^2+4y+1}}{4y^2}-4} \\ &= \frac{2y+2y\sqrt{16y^2+4y+1}+4y^2}{16y^2+4y+2+2\sqrt{16y^2+4y+1}-16y^2} \\ &= \frac{y+y\sqrt{16y^2+4y+1}+2y^2}{2y+1+\sqrt{16y^2+4y+1}} \\ &= \frac{y(2y+1+\sqrt{16y^2+4y+1})}{2y+1+\sqrt{16y^2+4y+1}} \\ &= y. \end{aligned}$$

The proof is complete. $\square$

Comment. This question is more difficult than I initially expected; the formulae are *not nice*. The important takeaway is the structure of a function range proof. Suppose you are proving that for a function $f : A \to B$, the range of f is some set $D \subseteq B$:

- First, prove that $f(A) \subseteq D$. Let $y \in f(A)$, so by definition $f(x) = y$ for some $x \in A$. Use this to show that $y \in D$.
- Then, show that $D \subseteq f(A)$. Let $y \in D$, and find an $x \in A$ such that $f(x) = y$. Sometimes this x can be found by "inverting" the formula for f (if f is defined via formula), but sometimes this approach does not work and you will need to guess the x. Either way, once you find the $x \in A$ (you might have to verify that $x \in A$ indeed), you will need to show that $f(x) = y$.

Problem 4
Find an example of a function $f : A \to B$ and sets $C, D \subseteq A$ such that $f(C \cap D) \neq f(C) \cap f(D)$.
Solution
Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Let $C = [-2, 0], D = [0, 2]$, both of which are subsets of our domain $\mathbb{R}$. We have $f(C) = [0, 4]$, $f(D) = [0, 4]$.[a]

[a] I will provide a proof that $f(C) = [0, 4]$; the proof for $f(D) = [0, 4]$ is omitted. $f(C) \subseteq [0, 4]$: let $y \in f(C)$. Then there exists $x \in C$ such that $y = x^2$. Certainly $x^2 \geq 0$. Also, $x \in C = [-2, 0]$ implies $-2 \leq x \leq 0$, which means $|x| \leq 2$. Thus $x^2 = |x|^2 \leq 2^2 = 4$. Thus $y = x^2 \in [0, 4]$

Problem 5
Let A be the "set of all sets". Define $S = \{X \in A : X \notin X\}$. Identify a logical contradiction, assuming the existence of S.[a]
Solution
Exactly one of $S \in S$ or $S \notin S$ must be true. If $S \in S$, then $S \in \{X \in A : X \notin X\}$, the set of all sets that don't contain themselves, so $S \notin S$. However, we've assumed that $S \in S$, producing a logical contradiction. Instead, assume $S \notin S$. But since $S \in A$ (as S is a set) and $S \notin S$, so

$S \in \{X \in A : X \notin X\}$. In other words $S \in S$, again producing a contradiction.

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[a] This contradiction is called **Russell's Paradox**, which arises due to our informal and loose definition of a "set" as a collection of distinct objects. In this course, you won't have to worry about such malicious examples of "sets", but it may be helpful to remember that MAT102 presents a simplified but workable version of set theory.

One way to avoid Russell's Paradox is to use a more formal axiomatic definition of "set", such as the **Zermelo-Frankel Axioms (ZF)**. Under the ZF axioms, sets can't be a members of themselves ($X \notin X$ for all sets X).