Problem 1

1. Provide the definition of a field. List out and name all the field axioms.[a]
2. List out all the fields you know.

[a]Consult Definition 5.13 in the Course Notes if needed.

Problem 2

Let $F = \{0, 1, a\}$. Complete the following addition and multiplication tables for F.

+	0	1	a
0			
1			
a			

·	0	1	a
0			
1			
a			

Problem 3

Let F be a field, and $a, b \in F$.

1. Suppose $ab = 0$. Show that $a = 0$ or $b = 0$. [a] You may use Claim 2.3.2.
2. Show that $a^2 - b^2 = (a + b)(a - b)$.
3. Suppose $a^2 = b^2$. Show that $a = -b$ or $a = b$.

[a]This is known as the **zero-product property**.

Problem 4

Define $F = \mathbb{R} \times \mathbb{R}$. We define addition $+$ and multiplication $\cdot$ over F in the following way:

- $(a, b) + (c, d) = (a + b, c + d)$ (where $a + b$ and $c + d$ is just addition of real numbers).
- $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$ (where again the operations are over real numbers).

1. Show that F is a field. *Hint: The multiplicative inverse of* (a, b) *is* $\left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right)$.
2. Show that there is $(a, b) \in F$ such that $(a, b) \cdot (a, b) = -1$ (where -1 is the additive inverse of the additive identity 1 in F).

Comment. F is the *complex numbers*; (a, b) corresponds with $a + bi$. This problem asks you to show that the complex numbers form a field.

Problem 5

Suppose $F \subseteq \mathbb{R}$ is a field with addition and multiplication inherited from the real numbers.[a]

1. Show that $\mathbb{N} \subseteq F$.
2. Show that $\mathbb{Z} \subseteq F$.
3. Show that $\mathbb{Q} \subseteq F$.[b]

[a]In other words, to add or multiply any two elements $a, b \in F$, treat a and b as real numbers.
[b]This is Exercise 2.5.52 from the Course Notes.