Problem 1

Find the range of each of the following functions.

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{1+2^{-x}}$.
2. $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = (-1)^n$.
3. $f : \mathbb{N} \times \mathbb{Z} \to \mathbb{R}$, $f(n, m) = n + m - 2$.

Solution

1. $f(\mathbb{R}) = (0, 1)$:[a]

   - $f(\mathbb{R}) \subseteq (0, 1)$: Suppose $y \in f(\mathbb{R})$. Then there is some $x \in \mathbb{R}$ such that $y = \frac{1}{1+2^{-x}}$. Notice that $y > 0$ since $1 > 0$ and $1 + 2^{-x} > 0$ (a ratio of two positive numbers is positive). Also, $y < 1$: since $2^{-x} > 0$, we have $1 + 2^{-x} > 1$, or $1 > \frac{1}{1+2^{-x}} = y$. Thus $y \in (0, 1)$.
   - $(0, 1) \subseteq f(\mathbb{R})$: Suppose $y \in (0, 1)$. We want to find $x \in \mathbb{R}$ such that $y = \frac{1}{1+2^{-x}}$.

     Rough Work

     $$\begin{aligned} y &= \frac{1}{1+2^{-x}} \\ y + 2^{-x}y &= 1 \\ 2^{-x} &= \frac{1-y}{y} \\ -x &= \log_2\left(\frac{1-y}{y}\right) \\ x &= -\log_2\left(\frac{1-y}{y}\right) \end{aligned}$$

     Let $x = -\log_2\left(\frac{1-y}{y}\right)$ (which is defined since $y \in (0, 1)$ implies $y \neq 0$ and $\frac{1-y}{y} > 0$). Clearly $x \in \mathbb{R}$. We have

     $$f(x) = \frac{1}{1 + 2^{\log_2\left(\frac{1-y}{y}\right)}} = \frac{1}{1 + \frac{1-y}{y}} = \frac{y}{y + 1 - y} = y.$$

     Thus $y \in f(\mathbb{R})$.

2. $f(\mathbb{N}) = \{-1, 1\}$:[b]

   - $f(\mathbb{N}) = \{-1, 1\}$: Let $y \in f(\mathbb{N})$. Then $y = (-1)^n$ for some $n \in \mathbb{N}$. If n is even then $y = 1$; if n is odd then $y = -1$. In either case, $y \in \{-1, 1\}$.
   - $\{-1, 1\} \subseteq f(\mathbb{N})$: Let $y \in \{-1, 1\}$. If $y = -1$, then $y = (-1)^1 = f(1)$; if $y = 1$, then $y = (-1)^2 = f(2)$. In either case, $y \in f(\mathbb{N})$.[c]

3. $f(\mathbb{N} \times \mathbb{Z}) = \mathbb{Z}$:

   - $f(\mathbb{N} \times \mathbb{Z}) = \mathbb{Z}$: Let $y \in f(\mathbb{N} \times \mathbb{Z})$. Then there exists $(n, m) \in \mathbb{N} \times \mathbb{Z}$ such that $y = n + m - 2$. So y is the sum of three integers (n, m, -2), hence $y \in \mathbb{Z}$.
   - $\mathbb{Z} \subseteq f(\mathbb{N} \times \mathbb{Z})$: Let $y \in \mathbb{Z}$. Then $y = 2 + y - 2 = f(2, y)$, so $y \in f(\mathbb{N} \times \mathbb{Z})$.

[a] I found this range by using a graphing calculator; on a quiz/test you may choose to use an analytical approach instead, or "brute force" to graph the function by hand.

[b] For functions like these where you may not be able to graph them, you'll just have to make an educated guess about the range.

[c] Notice that in this proof, we only need to come up with *one* input $x \in \mathbb{N}$ with $y = f(x)$, to show that $y \in f(\mathbb{N})$. Recall that $y \in f(\mathbb{N})$ if and only if there is *some* $x \in \mathbb{N}$ such that $f(x) = y$.

Problem 2
Let A, B, C be sets. Suppose $A \setminus C = B \setminus C$.

1. Give an example of sets A, B, C satisfying the above such that $A \neq B$.
2. Suppose furthermore that $A \cap C = B \cap C$. Show that $A = B$.

Solution

1. $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$ gives us $A \setminus C = \{1\} = B \setminus C$ yet $A \neq B$.
2.
   - ($A \subseteq B$): suppose $x \in A$. Now either $x \in C$ or $x \notin C$. If $x \in C$, then $x \in A \cap C = B \cap C$, so $x \in B$. If $x \notin C$, then $x \in A \setminus C = B \setminus C$, so $x \in B$.
   - ($B \subseteq A$): this is a symmetric argument.

Problem 3
Let $f : A \to B$ be a function, and $C \subseteq B$. We define the **preimage** of the set C as

$$f^{-1}(C) = \{x \in A : f(x) \in C\}.$$

(Note that "f^{-1}" should not be confused with the *inverse* of f; the inverse of f might not exist.)

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Find $f^{-1}((1, 4])$.
2. Ignoring the context of the previous subquestion, suppose $C, D \subseteq B$. Show that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
3. Is it true that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$? Give a proof or counterexample.

Solution

1. $f^{-1}((1, 4]) = [-2, -1) \cup (1, 2]$:
   - $f^{-1}((1, 4]) \subseteq [-2, -1) \cup (1, 2]$: Let $x \in f^{-1}((1, 4])$. Then by definition, $f(x) \in (1, 4]$, or $1 < x^2 \leq 4$. $1 < x^2$ gives $x \in (-\infty, -1) \cup (1, \infty)$, while $x^2 \leq 4$ gives $x \in (-2, 2)$. Thus $x \in ((-\infty, -1) \cup (1, \infty)) \cap (-2, 2) = [-2, -1) \cup (1, 2]$.
   - $[-2, -1) \cup (1, 2] \subseteq f^{-1}((1, 4])$: Let $x \in [-2, -1) \cup (1, 2]$. If $x \in [-2, -1)$, then $-2 \leq x < -1$, so $1 < x^2 \leq 4$; if $x \in (1, 2]$, then $1 < x \leq 2$ so $1 < x^2 \leq 4$. In either case, $x^2 \in (1, 4]$, so $f(x) \in (1, 4]$ or $x \in f^{-1}((1, 4])$.
2.
   - $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$: Suppose $x \in f^{-1}(C \cap D)$. Then $f(x) \in C \cap D$, so $f(x) \in C$ and $f(x) \in D$, respectively giving us $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. So $x \in f^{-1}(C) \cap f^{-1}(D)$.
   - $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$: Reverse the previous argument.
3. It is true; we give a proof.

- $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$: Suppose $x \in f^{-1}(C \cup D)$. Then $f(x) \in C \cup D$, so $f(x) \in C$ or $f(x) \in D$, respectively giving us $x \in f^{-1}(C)$ or $x \in f^{-1}(D)$. So $x \in f^{-1}(C) \cup f^{-1}(D)$.
- $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$: Reverse the previous argument.