Problem 1
Determine which of the following expressions are mathematical statements. Of those that are statements, determine whether they are true or false.

1. $(\forall x \in \mathbb{R})(x > y)$.
2. $(\forall x \in \mathbb{R})(x > \pi)$.
3. $(\exists x \in \emptyset)(x = x)$.
4. $(\forall x \in \emptyset)(x = x)$.
5. $\emptyset \Leftrightarrow (\forall x \in \mathbb{R})(x \notin S)$ (where S is some predetermined set).
6. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 > y)$.
7. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 > y)$.
8. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 > y)$.
9. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 > y)$.

Solution

1. This is not a statement; y is not yet defined.
2. This is a false statement, because not all real numbers are greater than π; for example, 0 is not greater than π.
3. This is a false statement; there is no $x \in \emptyset$ in the first place.
4. This is a vacuously true statement.
5. This is not a statement: $\Leftrightarrow$ is a logical connective; both sides of $\Leftrightarrow$ must be statements. However, "$\emptyset$" is not a statement, since it is not making a mathematical assertion.
6. This is a false statement; not all real x and y have to satisfy $x^2 > y$ (for example $x = 0$ and $y = 1$).
7. This is a true statement; there exist real x and y that satisfy $x^2 > y$ (such as $x = 1$ and $y = 0$).
8. This is a false statement; there is no x such that x^2 is greater than any real number y.
9. This is a true statement; given any $x \in \mathbb{R}$, we have $x^2 > -1$ for example.

Problem 2
For each valid mathematical statement in Problem 1, write out its negation.
Solution

2. $(\exists x \in \mathbb{R})(x \leq \pi)$.
3. $(\forall x \in \emptyset)(x \neq x)$.
4. $(\exists x \in \emptyset)(x \neq x)$.
6. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 \leq y)$.
7. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 \leq y)$.
8. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x^2 \leq y)$.
9. $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x^2 \leq y)$.

Problem 3
Let P, Q, R be statements. Use a truth table to show "$P \Rightarrow ((Q \wedge R) \vee (\neg Q \wedge \neg R))$" is logically equivalent to "$\neg P \vee (Q \Leftrightarrow R)$".
Solution

P	Q	R	$P \Rightarrow ((Q \wedge R) \vee (\neg Q \wedge \neg R))$	$\neg P \vee (Q \Leftrightarrow R)$
F	F	F	T	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	T	T
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T

As seen, the truth values of "$P \Rightarrow ((Q \wedge R) \vee (\neg Q \wedge \neg R))$" and "$\neg P \vee (Q \Leftrightarrow R)$" always match.

Problem 4
Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and $c, L \in \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

(ϵ and δ are the Greek letters Epsilon and Delta respectively.)

1. Let $f(x) = x$. Show that
$$\lim_{x\to 0} f(x) = 0.$$
2. Let $f(x) = ax$, where $a \in \mathbb{R}$ is some constant. Show that
$$\lim_{x\to 1} f(x) = a.$$
3. (Harder) Let $f(x) = x^2$. Show that
$$\lim_{x\to 2} f(x) = 4.$$

Solution

1. Let $\epsilon > 0$. Set $\delta = \epsilon$ (which is > 0). Let $x \in \mathbb{R}$. Suppose $0 < |x - 0| < \delta$. Then
$$|f(x) - 0| = |x - 0| < \delta = \epsilon.$$
2. We split into cases.
   - $a = 0$: Let $\epsilon > 0$. Set $\delta = 1$ (or any positive number of your choice). Let $x \in \mathbb{R}$. Suppose $0 < |x - 1| < \delta$. Then
   $$|f(x) - a| = |0x - 0| = 0 < \epsilon.$$
   Comment. It doesn't matter what you set δ to here, as $|f(x) - a| < \epsilon$ holds regardless.
   - $a \neq 0$: Let $\epsilon > 0$. Set $\delta = \frac{1}{|a|}\epsilon$. Let $x \in \mathbb{R}$. Suppose $0 < |x - 1| < \delta$. Then
   $$|f(x) - a| = |ax - a| = |a(x - 1)| = |a||x - 1| < |a|\delta = \epsilon.$$

3. Let $\epsilon > 0$. Set $\delta = \min\{1, \frac{\epsilon}{5}\}$ (so that $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{5}$). Suppose $0 < |x-2| < \delta$. Observe that

$$\begin{aligned} &|x-2| < \delta \\ \Rightarrow &|x-2| < 1 \\ \Rightarrow &-1 < x-2 < 1 \\ \Rightarrow &3 < x+2 < 5 \\ \Rightarrow &-5 < x+2 < 5 \\ \Rightarrow &|x+2| < 5. \end{aligned} \tag{1}$$

Thus

$$\begin{aligned} |f(x)-4| &= |x^2-4| \\ &= |(x+2)(x-2)| \\ &= |x+2||x-2| \\ &< 5|x-2| && \text{by (1)} \\ &< 5\delta \\ &\leq 5\frac{\epsilon}{5} \\ &= \epsilon. \end{aligned}$$