Problem 1

Construct a truth table for each of the following predicates. Which of them are logically equivalent?

(a) $Q \Leftrightarrow (P \vee Q)$.

(b) $P \vee \neg Q$.

(c) $Q \Rightarrow P$.

(d) $(P \vee \neg P) \wedge (Q \Leftrightarrow (Q \wedge \neg Q))$.

Solution

P	Q	$Q \Leftrightarrow (P \vee Q)$	$P \vee \neg Q$	$Q \Rightarrow P$	$(P \vee \neg P) \wedge (Q \Leftrightarrow (Q \wedge \neg Q))$
T	T	T	T	T	F
T	F	F	T	T	T
F	T	F	F	F	F
F	F	T	T	T	T

Problem 2

Show that the polynomial

$$p(x) = 5x^5 - 3x^3 + 1$$

has no rational roots. *Hint: See example 3.6.3 in the Course Notes.*

Solution

We use proof by contradiction. Suppose $p(x)$ had a rational root, say $x = \frac{p}{q}$ with $p, q \in \mathbb{Z}, q \neq 0$. We assume that $\frac{p}{q}$ is in lowest terms (as every rational number has a lowest terms representation). Then

$$5\left(\frac{p}{q}\right)^5 - 3\left(\frac{p}{q}\right)^3 + 1 = 0.$$

Multiplying both sides by q^5, we have

$$5p^5 - 3p^3q^2 + q^5 = 0.$$

Since p is an integer, it is either odd or even, and the same can be said about q. We split into cases.

- p is odd, q is odd: Then $5p^5$, $-3p^3q^2$, q^5 are all odd, and since the sum of three odd numbers is odd, $5p^5 - 3p^3q^2 + q^5$ is odd. However 0 is even, which contradicts the equality (an odd integer can't equal an even integer).

- p is even, q is odd: Then $5p^5$ and $-3p^3q^2$ are both even, while q^5 is odd, which means $5p^5-3p^3q^2+q^5$ is odd. Again, this contradicts the equality as 0 is even.

- p is odd, q is even: Then $5p^5$ is odd, while $-3p^3q^2$ and q^5 are both even, which means $5p^5-3p^3q^2+q^5$ is odd. Again, this contradicts the equality as 0 is even.

- p is even, q is even: This contradicts our assumption that $\frac{p}{q}$ is in lowest terms.

In all cases we get a contradiction, showing that our assumption on the existence of a rational root for $p(x)$ is false.

Problem 3
Let $S = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

1. List all subsets of S. (How many subsets are there?)
2. Determine which of the following statements are true.
   - $\emptyset \subseteq S$.
   - $\emptyset \in S$.
   - $\emptyset \subseteq \{S\}$.
   - $\emptyset \in \{S\}$.
   - $\{\emptyset\} \subseteq S$.
   - $\{\emptyset\} \in S$.
   - $\{\{\emptyset\}\} \subseteq S$.
   - $\{\{\emptyset\}\} \in S$.

Solution

1. The subsets of S are:
   - $\emptyset$;
   - $\{\emptyset\}$;
   - $\{\{\emptyset\}\}$;
   - $\{\{\emptyset, \{\emptyset\}\}\}$;
   - $\{\emptyset, \{\emptyset\}\}$;
   - $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$;
   - $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$;
   - $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.
2. - True.
   - True.
   - True.
   - False.
   - True.
   - True.
   - True.
   - False.

Problem 4
Prove that $3^{2^n} - 1$ is divisible by 10 for all $n \in \mathbb{N}, n \geq 2$.
Solution
We prove by induction. Let $P(n)$ be the predicate "$3^{2^n} - 1$ is divisible by 10", where n is a natural number. We are to prove $P(2)$ (since the question asks us to prove the statement only for $n \geq 2$), and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{N}, k \geq 2$.

- Base case ($n = 2$): $3^{2^2} - 1 = 80$ which is indeed divisible by 10, verifying the base case $P(2)$.
- Induction step: Suppose $P(k)$ is true for some $k \in \mathbb{N}, k \geq 2$. Then $3^{2^k} - 1$ is divisible by 10. We want to show that $P(k+1)$ is true, i.e. $3^{2^{k+1}} - 1$ is divisible by 10. We have

$$\begin{aligned} 3^{2^{k+1}} - 1 &= 3^{2^k \cdot 2} - 1 \\ &= \left(3^{2^k}\right)^2 - 1 \\ &= (3^{2^k} - 1)(3^{2^k} + 1) && \text{(Difference of squares)} \end{aligned}$$

  As $3^{2^k} - 1$ is divisible by 10 by induction hypothesis, so is $(3^{2^k} - 1)(3^{2^k} + 1)$. Thus $3^{2^{k+1}} - 1$ is divisible by 10, completing the induction step.

By the principle of mathematical induction, $P(n)$ holds true for all $n \in \mathbb{N}, n \geq 2$ as needed.