Problem 1
Find an $M \in \mathbb{R}$ such that

$$\left|\frac{xy - y^2 + 4}{x^2 + y^2 + 1}\right| \leq M$$

for all $x, y > 0$.

Problem 2
Using the triangle inequality, prove that for any $x, y \in \mathbb{R}$,

$$|x - y| \geq |x| - |y|.$$

Hint: Rearrange the above inequality.

Problem 3
Define the sequence (a_n) recursively:

$$a_0 = 5, \quad a_{n+1} = 2a_n + 5 \text{ (for } n \geq 0).$$

Prove, by induction, that

$$a_n = 5 \cdot (2^{n+1} - 1).$$

Problem 4
(*Exercise 4.6.29*) Let x be a nonzero real number, such that $x + \frac{1}{x}$ is an integer. Prove that for all $n \in \mathbb{N}$, the number $x^n + \frac{1}{x^n}$ is also an integer.

Hint: You will need two base cases. For the induction step, consider $(x + \frac{1}{x})(x^n + \frac{1}{x^n})$.