Problem 1
Find an $M \in \mathbb{R}$ such that

$$\left|\frac{xy - y^2 + 4}{x^2 + y^2 + 1}\right| \leq M$$

for all $x, y > 0$.
Solution
We have, via the triangle inequality,

$$\begin{aligned}
&\left|\frac{xy - y^2 + 4}{x^2 + y^2 + 1}\right| \\
\leq &\left|\frac{xy}{x^2 + y^2 + 1}\right| + \left|\frac{-y^2}{x^2 + y^2 + 1}\right| + \left|\frac{4}{x^2 + y^2 + 1}\right|.
\end{aligned}$$

Now, to bound $\left|\frac{xy}{x^2 + y^2 + 1}\right|$, notice that $(x - y)^2 \geq 0$ always, which gives us

$$\begin{aligned}
&(x - y)^2 \geq 0 \\
\Rightarrow &x^2 - 2xy + y^2 \geq 0 \\
\Rightarrow &x^2 + y^2 \geq 2xy \\
\Rightarrow &x^2 + y^2 > xy && \text{(since } x, y > 0\text{, we have } 2xy > xy) \\
\Rightarrow &x^2 + y^2 + 1 > xy \\
\Rightarrow &1 > \frac{xy}{x^2 + y^2 + 1} && \text{(since } x^2 + y^2 + 1 > 0) \\
\Rightarrow &1 > \left|\frac{xy}{x^2 + y^2 + 1}\right| && \text{(since } a = |a| \text{ if } a > 0).
\end{aligned}$$

To bound $\left|\frac{-y^2}{x^2 + y^2 + 1}\right|$, we have

$$\left|\frac{-y^2}{x^2 + y^2 + 1}\right| = \left|\frac{y^2}{x^2 + y^2 + 1}\right| = \frac{y^2}{x^2 + y^2 + 1} \leq \frac{y^2}{y^2} = 1.$$

Finally, to bound $\left|\frac{4}{x^2 + y^2 + 1}\right|$, we have

$$\left|\frac{4}{x^2 + y^2 + 1}\right| = \frac{4}{x^2 + y^2 + 1} \leq \frac{4}{1} = 4.$$

Combining these three bounds,

$$\begin{aligned}
&\left|\frac{xy - y^2 + 4}{x^2 + y^2 + 1}\right| \\
\leq &\left|\frac{xy}{x^2 + y^2 + 1}\right| + \left|\frac{-y^2}{x^2 + y^2 + 1}\right| + \left|\frac{4}{x^2 + y^2 + 1}\right| \\
\leq &1 + 1 + 4 = 6.
\end{aligned}$$

So choosing $M = 6$ would work.

Problem 2
Using the triangle inequality, prove that for any $x, y \in \mathbb{R}$,

$$|x - y| \geq |x| - |y|.$$

Hint: Rearrange the above inequality.
Solution
We have

$$\begin{aligned} |x| &= |(x - y) + y| \\ &\leq |x - y| + |y| && \text{(triangle inequality)} \end{aligned}$$

and rearranging,

$$|x - y| \leq |x - y|$$

as needed.

Problem 3
Define the sequence (a_n) recursively:

$$a_0 = 5, \quad a_{n+1} = 2a_n + 5 \text{ (for } n \geq 0).$$

Prove, by induction, that

$$a_n = 5 \cdot (2^{n+1} - 1).$$

Solution
Let $P(n)$ be the predicate "$a_n = 5 \cdot (2^{n+1} - 1)$" for $n \in \mathbb{Z}, n \geq 0$. We show $P(n)$ is true for all $n \geq 0$ via induction.

- Base case ($P(0)$): we have $a_0 = 5$ and $5 \cdot (2^{0+1} - 1) = 5 \cdot (2 - 1) = 5$ as needed.
- Induction step: Assume that $P(k)$ is true for some $k \geq 0$. We show $P(k+1)$ is true. Notice that

$$\begin{aligned} a_{k+1} &= 2a_k + 5 && \text{(by definition of the sequence } a_n) \\ &= 2(5 \cdot (2^{k+1} - 1)) + 5 && \text{(by induction hypothesis } P(k)) \\ &= 10 \cdot (2^{k+1} - 1) + 5 \\ &= 5 \cdot (2 \cdot (2^{k+1} - 1) + 1) \\ &= 5 \cdot (2^{k+2} - 2 + 1) \\ &= 5 \cdot (2^{k+2} - 1). \end{aligned}$$

  This shows $P(k+1)$.

By the principle of mathematical induction, $P(n)$ is true for all $n \geq 0$ as needed.

Problem 4
(*Exercise 4.6.29*) Let x be a nonzero real number, such that $x + \frac{1}{x}$ is an integer. Prove that for all $n \in \mathbb{N}$, the number $x^n + \frac{1}{x^n}$ is also an integer.

Hint: You will need two base cases. For the induction step, consider $(x + \frac{1}{x})(x^n + \frac{1}{x^n})$.

Solution
Supposing $x + \frac{1}{x} \in \mathbb{Z}$, let $P(n)$ be the predicate "$x^n + \frac{1}{x^n} \in \mathbb{Z}$" for all $n \in \mathbb{N}$. We show $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction.

- Base cases:
  - $P(1)$: $x + \frac{1}{x}$ is indeed an integer, by assumption in the question.
  - $P(2)$: We notice that
  $$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2.$$
  Since $x + \frac{1}{x} \in \mathbb{Z}$, so is $\left(x + \frac{1}{x}\right)^2 \in \mathbb{Z}$. Thus $x^2 + \frac{1}{x^2}$ is equal to the difference of two integers, which shows $x^2 + \frac{1}{x^2} \in \mathbb{Z}$.

- Induction step: Suppose that $P(1), \ldots, P(k)$ are all true, for some $k \geq 2$. In particular, since $k \geq 2$, we know that $P(k)$ and $P(k-1)$ are true. We have
$$\left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) = x^{k+1} + x^{k-1} + \frac{1}{x^{k-1}} + \frac{1}{x^{k+1}}.$$
Rearranging,
$$x^{k+1} + \frac{1}{x^{k+1}} = \left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right).$$
The assumption in the question tells us $\left(x + \frac{1}{x}\right) \in \mathbb{Z}$, while $P(k)$ in our induction hypothesis says that $\left(x^k + \frac{1}{x^k}\right) \in \mathbb{Z}$, and $P(k-1)$ in our induction hypothesis says that $\left(x^{k-1} + \frac{1}{x^{k-1}}\right) \in \mathbb{Z}$. Thus $x^{k+1} + \frac{1}{x^{k+1}}$ is the difference of two integers, which shows $\left(x^{k-1} + \frac{1}{x^{k-1}}\right) \in \mathbb{Z}$, completing the induction step.

Using the principle of strong mathematical induction, we have shown that $P(n)$ holds for all $n \in \mathbb{N}$ as needed.

Comment. The reason why we needed two base cases is because in our induction step, we needed the previous two induction hypothesis $P(k-1)$ and $P(k)$ to prove $P(k+1)$. If we only had one base case, then we would be unable to prove $P(2)$, as that would require both $P(0)$ and $P(1)$, but we haven't proven or even defined $P(0)$.