Problem 1
Decide whether the following functions are injective, surjective, bijective, or neither.

- $f : \mathbb{R} \to \mathbb{Z}, f(x) = \lfloor x \rfloor$.[a]
- $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$, $f(x, y) = |x - y|$.
- $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$, $f(n) = (2n + 3, 4n - 2)$.
- $f : \mathbb{Q} \to \mathbb{N}$, $f(\frac{p}{q}) = p^2$ ($\frac{p}{q}$ in lowest terms).
- $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [0, 1]$, $f(x) = \cos(x)$.
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x| + x$.

[a] $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.

Problem 2
A function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* if for all $x, y \in \mathbb{R}$, $x < y \Rightarrow f(x) < f(y)$.

- Show that if f is increasing, then f is injective.
- Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be *decreasing*. Is it still true that if f is decreasing, then it is injective?

Problem 3
Suppose $P(n)$ is a statement for all $n \in \mathbb{N}$. For each of the following induction schema, determine for which $n \in \mathbb{N}$ the schema proves $P(n)$.

ex. $P(2)$, and $P(k) \Rightarrow P(k + 1)$ for $k \in \mathbb{N}$. This schema proves $P(n)$ for $n \in \mathbb{N}, n \geq 2$.

- $P(1)$, and $P(k) \Rightarrow P(k + 2)$ for $k \in \mathbb{N}$.
- $P(1)$, and $P(k), P(k + 1) \Rightarrow P(k + 2)$ for $k \in \mathbb{N}$.
- $P(2)$, and $P(k) \Rightarrow P(k + 2)$ for $k \in \mathbb{N}$, and $P(k) \Rightarrow P(\frac{k}{2})$ for even $k \in \mathbb{N}$.
- $P(2)$, and $P(k) \Rightarrow P(k + 5)$ for $k \in \mathbb{N}$, and $P(k) \Rightarrow P(k - 3)$ for $k \in \mathbb{N}, k \geq 4$.

Problem 4
Prove, by induction, that

$$\sum_{i=0}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

for all integers $n \geq 0$.