Problem 1

Decide whether the following functions are injective, surjective, bijective, or neither.

- $f : \mathbb{R} \to \mathbb{Z}, f(x) = \lfloor x \rfloor$.[a]
- $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \cup \{0\}$, $f(x, y) = |x - y|$.
- $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$, $f(n) = (2n + 3, 4n - 2)$.
- $f : \mathbb{Q} \to \mathbb{N} \cup \{0\}$, $f(\frac{p}{q}) = p^2$ ($\frac{p}{q}$ in lowest terms).
- $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [0, 1]$, $f(x) = \cos(x)$.
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x| + x$.

Solution

- $f : \mathbb{R} \to \mathbb{Z}, f(x) = \lfloor x \rfloor$ is not injective, as $f(\pi) = f(3) = 3$ (yet $\pi \neq 3$). However, f is surjective, since its range $f(\mathbb{R})$ is equal to the codomain $\mathbb{Z}$: for any $y \in \mathbb{Z}$, we have $f(y) = y$, so $y \in f(\mathbb{R})$ (hence $\mathbb{Z} \subseteq f(\mathbb{R})$), and the other inclusion $f(\mathbb{R}) \subseteq \mathbb{Z}$ is immediate.
- $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{N}\cup\{0\}$, $f(x, y) = |x-y|$ is not injective, as $f(0, 1) = f(0, -1) = 1$ (yet $(0, 1) \neq (0, -1)$). However, f is surjective: for any $z \in \mathbb{N} \cup \{0\}$, we have $f(z, 0) = |z - 0| = z$ since z is nonnegative.
- $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$, $f(n) = (2n + 3, 4n - 2)$ is injective: if $m, n \in \mathbb{N}$ and $m \neq n$, then either $m < n$ or $m > n$. If $m < n$, then $2m + 3 < 2n + 3$ (so $2m + 3 \neq 2n + 3$), which means $f(m) \neq f(n)$ as their first entries differ. A similar argument holds for $m > n$. f is not surjective however, as $(0, 1)$ is not in the range: $2n + 3$ (which is always odd) can never equal 0 (which is even).
- $f : \mathbb{Q} \to \mathbb{N} \cup \{0\}$, $f(\frac{p}{q}) = p^2$ ($\frac{p}{q}$ in lowest terms) is not injective, as $f(\frac{1}{2}) = f(\frac{1}{3}) = 1^2 = 1$. Nor is it surjective: 3 is not in the range, as f can only map to square numbers but 3 isn't a square number.
- $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [0, 1]$, $f(x) = \cos(x)$ is not injective, as $f(-\frac{\pi}{2}) = f(\frac{\pi}{2}) = 0$. f however is surjective, as for any $y \in [0, 1]$, $\arccos(y)$ is defined and in $[0, \frac{\pi}{2}]$, with $f(\arccos(y)) = y$.
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x| + x$ is not injective, as $f(0) = f(-1) = 0$. Neither is it surjective, as f can only map to nonnegative numbers: for $x \geq 0$ we have $f(x) = x + x = 2x$, while for $x \leq 0$ we have $f(x) = -x + x = 0$. So -1 for example is not in the range.

[a] $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.

Problem 2

A function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* if for all $x, y \in \mathbb{R}$, $x < y \Rightarrow f(x) < f(y)$.

- Show that if f is increasing, then f is injective.
- Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be *decreasing*. Is it still true that if f is decreasing, then it is injective?

Solution

- Suppose f is increasing, and $x, y \in \mathbb{R}$, with $x \neq y$. Either $x < y$ or $x > y$. If $x < y$, then $f(x) < f(y)$ (by definition of increasing), so $f(x) \neq f(y)$ as needed. If $x > y$, then $f(y) > f(x)$ so again $f(x) \neq f(y)$ as needed.
- A function $f : \mathbb{R} \to \mathbb{R}$ is *decreasing* if for all $x, y \in \mathbb{R}$, $x < y \Rightarrow f(x) > f(y)$. Indeed, any decreasing function is still injective, by modifying the proof above.

Problem 3
Suppose $P(n)$ is a statement for all $n \in \mathbb{N}$. For each of the following induction schema, determine for which $n \in \mathbb{N}$ the schema proves $P(n)$.

ex. $P(2)$, and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{N}$. This schema proves $P(n)$ for $n \in \mathbb{N}, n \geq 2$.

- $P(1)$, and $P(k) \Rightarrow P(k+2)$ for $k \in \mathbb{N}$.
- $P(1)$, and $[P(k) \wedge P(k+1)] \Rightarrow P(k+2)$ for $k \in \mathbb{N}$.
- $P(2)$, and $P(k) \Rightarrow P(k+2)$ for $k \in \mathbb{N}$, and $P(k) \Rightarrow P(\frac{k}{2})$ for even $k \in \mathbb{N}$.
- $P(2)$, and $P(k) \Rightarrow P(k+5)$ for $k \in \mathbb{N}$, and $P(k) \Rightarrow P(k-3)$ for $k \in \mathbb{N}, k \geq 4$.

Solution

- This proves $P(n)$ for odd n only.
- This proves only $P(1)$: The rule "$[P(k) \wedge P(k+1)] \Rightarrow P(k+2)$ for $k \in \mathbb{N}$" requires us to know two consecutive $P(k)$ and $P(k+1)$, yet we only have one ($P(1)$) to start off. *We need more base cases!*
- This proves $P(n)$ for all $n \in \mathbb{N}$: notice that $P(2)$ and $P(k) \Rightarrow P(k+2)$ prove $P(n)$ for all even $n \in \mathbb{N}$. Now for odd $n \in \mathbb{N}$, since $2n$ is even, $P(2n)$ is proven; applying $P(k) \Rightarrow P(\frac{k}{2})$ we have $P(2n) \Rightarrow P(n)$ so $P(n)$ is proven as well.
- This proves $P(n)$ for all $n \in \mathbb{N}$ too. Notice that this scheme shows $P(k) \Rightarrow P(k-1)$ for $k \geq 2$, as $P(k) \Rightarrow P(k+5) \Rightarrow P(k+2) \Rightarrow P(k-1)$. Thus, given $P(2)$, we can prove $P(1)$. Now using the rule $P(k+5)$, we have $P(7)$, $P(12)$, $P(17)$, and so on; using the rule $P(k) \Rightarrow P(k-1)$ we can obtain $P(6), P(5), P(4), P(3)$ from $P(7)$, $P(11), P(10), P(9), P(8)$ from $P(12)$, and so on.

Problem 4
Prove, by induction, that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 0$. **Solution**
Define

$$P(n) : \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 0$. We prove $P(0)$, and $P(k) \Rightarrow P(k+1)$ for $k \geq 0$.

- $P(0)$: We have $\sum_{i=0}^{0} i^2 = 0^2 = 0$, and $\frac{0(0+1)(2\cdot 0+1)}{6} = 0$.
- $P(k) \Rightarrow P(k+1)$: suppose $P(k)$ is true, i.e.

  $$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

  We would like to prove $P(k+1)$, i.e.

  $$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

We have

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \left(\sum_{i=0}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{(Induction hypothesis)} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)[(k+2)(2k+3)]}{6}.
\end{aligned}$$

which completes the proof.