Problem 1

For each of the following, find a bijection from A to B.

1. $A = \mathbb{N}$, $B = \mathbb{N} \setminus \{1, 3\}$.
2. $A = \mathbb{N}$, $B = \mathbb{Z}$.
3. $A = (0, \infty)$, $B = \mathbb{R}$.
4. $A = \mathbb{R}$, $B = (-1, 1)$.

Solution

1. $f : \mathbb{N} \to \mathbb{N} \setminus \{1, 3\}$, $f(n) = \begin{cases} 2 & n = 1 \\ n + 2 & n > 1. \end{cases}$
2. $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} 0 & n = 1 \\ \frac{n}{2} & n > 1, n \text{ is even} \\ -\frac{n-1}{2} & n > 1, n \text{ is odd.} \end{cases}$
3. $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln(x)$.
4. $f : \mathbb{R} \to (-1, 1)$, $f(x) = \frac{1}{\pi} \arctan(x) \ + \ \frac{1}{2}$

Problem 2

Which of the following sets are not countable?

1. $\mathbb{R} \setminus \mathbb{Q}$.
2. $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.
3. $\mathcal{P}(\mathbb{N})$.
4. $\{f : f \text{ is a function with domain } \mathbb{N} \text{ and codomain } \mathbb{Q}\}$.

Solution

1. $\mathbb{R} \setminus \mathbb{Q}$ is not countable: for the sake of contradiction, suppose $\mathbb{R} \setminus \mathbb{Q}$ were countable instead. We have $\mathbb{R} = \mathbb{Q} \setminus (\mathbb{R} \setminus \mathbb{Q})$, so $\mathbb{R}$ is the union of two disjoint countable sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. This implies $\mathbb{R}$ is countable, contrary to the fact that $\mathbb{R}$ is uncountable.
2. $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is countable: the Cartesian product of any two countable sets is countable, so $\mathbb{Q} \times \mathbb{Q}$ is countable. Now $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ can be written as $(\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}$[a] which is again the Cartesian product of two countable sets, hence countable.
3. $\mathcal{P}(\mathbb{N})$ is uncountable by Cantor's theorem: $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$ (and a set is countable iff its cardinality is equal to that of $\mathbb{N}$).
4. $\{f : f \text{ is a function with domain } \mathbb{N} \text{ and codomain } \mathbb{Q}\}$ is not countable.[b] We use a diagonal argument.

   Suppose the set of functions from $\mathbb{N}$ to $\mathbb{Q}$ were countable. Then we can list *all* such functions $f_1, f_2, f_3, \ldots$. We will construct a function $g : \mathbb{N} \to \mathbb{Q}$ that is different from all the f_i's, which shows our enumeration is incomplete contrary to the fact that we just listed all such functions.

   Define
   $$g : \mathbb{N} \to \mathbb{Q}, g(n) = \begin{cases} 0 & f_n(n) \neq 0 \\ 1 & f_n(n) = 0. \end{cases}$$

   Notice that $g \neq f_n$ for any $n \in \mathbb{N}$, as we have chosen $g(n)$ so that $g(n) \neq f_n(n)$ (and two functions differ iff there is some input on which their output differs). Thus g is not in the list $f_1, f_2, f_3, \ldots$, which produces the contradiction we needed.

[a]Strictly speaking $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is *defined* to be $\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}$, as we have technically not defined what a triple Cartesian product is.

[b]Each individual function $f : \mathbb{N} \to \mathbb{Q}$ is countable (when functions are considered subsets of $\mathbb{N} \times \mathbb{Q}$). However, the collection of such functions is uncountable.

Problem 3

- Let $a < b$. Find a bijection from the interval $[a, b]$ to $[0, 1]$.
- Let $c < d$. Find a bijection from the interval $[0, 1]$ to $[c, d]$.
- Conclude that any two closed intervals have the same cardinality.

Solution

- I will use a linear function $f(x) = mx + c$ with endpoints $f(a) = 0$ and $f(b) = 1$. This gives us the system of equations

$$f(a) = ma + c = 0 \quad (1)$$
$$f(b) = mb + c = 1. \quad (2)$$

(2) gives $c = 1 - mb$. Substituting into (1),

$$ma + 1 - mb = 0 \Rightarrow m(a - b) = -1 \Rightarrow m = \frac{1}{b - a}.$$

Using this value of m in (1),

$$\frac{1}{b - a}a + c = 0 \Rightarrow c = -\frac{a}{b - a}.$$

Thus our bijection should be

$$f : [a, b] \to [0, 1], f(x) = \frac{1}{b - a}x - \frac{a}{b - a}.$$

Indeed, f is a bijection: f is injective since it is monotone increasing, and f is surjective as for any $y \in [0, 1]$, if we set $x = (b - a)y + a$, we have (since $b - a > 0$)

$$0 \le y \le 1 \Rightarrow 0 \le (b - a)y \le b - a \Rightarrow a \le (b - a)y + a \le b \Rightarrow a \le x \le b$$

so x is in the domain, and

$$f(x) = \frac{1}{b - a}((b - a)y + a) - \frac{a}{b - a} = y + \frac{a}{b - a} - \frac{a}{b - a} = y.$$

- I will again use a linear function $f(x) = mx + b$, and map $f(0) = c$, $f(1) = d$. Notice

$$f(0) = m \cdot 0 + b = b$$

and since $f(0) = c$, we have $b = c$. Now

$$f(1) = m \cdot 1 + b = m + c$$

and since $f(1) = d$, we have $m = d - c$. Thus our bijection should be

$$f : [0,1] \to [c,d], f(x) = (d-c)x + c.$$

The proof that f is a bijection is similar.

- Any two closed intervals $[a,b]$ and $[c,d]$ satisfy $|[a,b]| = |[0,1]| = |[c,d]|$, using (a).

Problem 4
Compute the power set of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. *Hint: You may find it easier to substitute $A = \emptyset$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$ before you start.*
Solution
As the hint suggests, let $A = \emptyset$, $B = \{\emptyset\}$, $C = \{\emptyset, \{\emptyset\}\}$. Then we are just computing the power set of $\{A, B, C\}$, which is

$$\{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}.$$

Substituting back, the power set should be

$$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}.$$

Problem 5
Let S be a set. Show that there is an injection from S to $\mathcal{P}(S)$.
Solution
Let $f : S \to \mathcal{P}(S)$ be defined as $f(s) = \{s\}$ for all $s \in S$. Notice that f is an injection: if $s, t \in S$ and $s \neq t$, then $\{s\} \neq \{t\}$.

Problem 6
Let U be the "set of all sets". Show that U cannot exist, using Cantor's theorem ($|S| < |P(S)|$ for any set S). *Hint: If U were a set, then what is $\mathcal{P}(U)$?*
Solution
Suppose U were a set. We claim $P(U) \subseteq U$. Indeed, for any $x \in P(U)$, by definition x is a subset of U. Hence x is a set, and since U is the "set of all sets" we have $x \in U$. Now since $P(U) \subseteq U$, there is an injection from $P(U)$ to U (just map everything in $P(U)$ to itself). Thus $|P(U)| \leq |U|$, contradicting Cantor's theorem.