MAT157 TUT9101

Welcome!

Info

Name: Paul Zhang

Email: pol.zhang@utoronto.ca

Office Hours: Wed 6-7pm, same room

Website (tutorial handouts will be posted here probably): sjorv.github.io

Discord: sjorv#0943

Interests: eating, pillow hugging, gaming, reading (text)books

Favourite food: sushi probably.

Please stop me if you have any questions! You may private message me on Zoom if preferred.

Problem 1

For each of the following statements, translate it into logical symbols. Try to also figure out whether the statements are true or false.

1. For every real number, there exists a real number such that their product is 1.
2. For every natural number, there exists a natural number greater than it.
3. If the product of two arbitrary integers is zero, one of them must be zero.
4. There is an integer that is its own square, and is not one or zero.
5. For any real numbers a, b, if $a - b$ is negative, then a must be greater than b.
6. An integer that is a perfect square and even must be divisible by 4.

(1) $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})[x\cdot y=1]$ False (if $x=0$, $\nexists y\in\mathbb{R}$ s.t. $x\cdot y=1$).

(2) $(\forall x\in\mathbb{N})(\exists y\in\mathbb{N})[y>x]$. True (for any $x\in\mathbb{N}$, if $y=x+1$, then $y>x$).

(3) $(\forall x\in\mathbb{Z})(\forall y\in\mathbb{Z})[x\cdot y=0 \Rightarrow (x=0)\vee(y=0)]$ True (zero-product property)

(4) $(\exists x\in\mathbb{Z})[(x=x^2)\wedge(x\neq 1)\wedge(x\neq 0)]$. $x=x^2 \Rightarrow x=0$ or $x=1$.

(5) $(\forall a,b\in\mathbb{R})[a-b<0 \Rightarrow a>b]$. False.

$(\forall a\in\mathbb{R})(\forall b\in\mathbb{R})$.

(6) $(\forall x\in\mathbb{Z})\left[\left((\exists a\in\mathbb{Z})[x=a^2]\wedge(\exists b\in\mathbb{Z})[x=2b]\right)\Rightarrow(\exists c\in\mathbb{Z})[x=4c]\right]$

x is a perfect square

x is even

True: if $x=a^2$ $(a\geq 0)$ and $x=2b$

$\sqrt{x}=a$ $\quad$ $\sqrt{x}=\sqrt{2}\sqrt{b}$

b must have a factor of 2.

Problem 2
For each of the following logic formulas, write down its negation in symbols. Determine whether it is true or its negation is true.

1. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})[m > n]$.
2. $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})[m > n]$.
3. $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})[m = 2n \Rightarrow (\forall \ell \in \mathbb{N})(m \neq 2\ell + 1)]$.
4. $(\forall q \in \mathbb{Q})(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})[q = m/n]$.
5. $(\exists r \in \mathbb{Q})(\forall s \in \mathbb{Q})[s^2 \leq 2 \Rightarrow r \geq s]$.
6. $(\exists \epsilon > 0)(\forall \delta > 0)(\forall x \in \mathbb{R})[0 < |x - 2| < \epsilon \Rightarrow |3x - 6| < \delta]$.[a]
7. $(\exists \epsilon > 0)(\forall \delta > 0)(\forall x \in \mathbb{R})[0 < |x - 2| < \epsilon \Rightarrow 0 < 2]$.

[a] ϵ and δ are the Greek letters Epsilon and Delta respectively. These two letters will reappear throughout the course and haunt you in your dreams.

(1) $\neg\boxed{(\forall n\in\mathbb{N})(\exists m\in\mathbb{N})[m>n]}$

$\Leftrightarrow (\exists n\in\mathbb{N})(\forall m\in\mathbb{N})[m\le n]$.

True (see problem 1, part 2)

* $\neg(\forall a\in A)\,p(a)$ is not $(\exists a\notin A)\,P(a)$

(2) $\neg\boxed{(\exists m\in\mathbb{N})(\forall n\in\mathbb{N})[m>n]}$ False (take $m=n$).

$\Leftrightarrow (\forall m\in\mathbb{N})(\exists n\in\mathbb{N})[m\le n]$.

(3) $\neg(\exists n\in\mathbb{N})(\forall m\in\mathbb{N})[m=2n\Rightarrow(\forall \ell\in\mathbb{N})(m\neq 2\ell+1)]$

$\Leftrightarrow (\forall n\in\mathbb{N})(\exists m\in\mathbb{N})[m=2n \wedge (\exists \ell\in\mathbb{N})(m=2\ell+1)]$

$\neg(P\Rightarrow Q)\Leftrightarrow P\wedge\neg Q$

True: if $m=2n$, then m is even. $2\ell+1$ is odd $\forall \ell$, so $(\forall \ell\in\mathbb{N})\ m\neq 2\ell+1$

P	Q	$P\Rightarrow Q$	$\neg(P\Rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

$P\wedge\neg Q$

(4) $\neg(\forall q\in\mathbb{Q})(\exists m\in\mathbb{N})(\exists n\in\mathbb{N})[q=\frac{m}{n}]$

$\Leftrightarrow (\exists q\in\mathbb{Q})(\forall m\in\mathbb{N})(\forall n\in\mathbb{N})[q\neq\frac{m}{n}]$.

False. if $q=-1$, then $\forall m,n\in\mathbb{N}$ $\frac{m}{n}>0$ so $\frac{m}{n}\neq q$.

(5) $\neg(\exists r\in\mathbb{Q})(\forall s\in\mathbb{Q})[s^2\le 2\Rightarrow r\ge s]$

$\Leftrightarrow (\forall r\in\mathbb{Q})(\exists s\in\mathbb{Q})[s^2\le 2 \wedge r<s]$.

True. If $r=2$

for any $s\in\mathbb{Q}$, if $s^2\le 2$ then $-\sqrt{2}\le s\le\sqrt{2}$.

so $s\le r$.

(6) $\neg(\exists \varepsilon>0)(\forall\delta>0)(\forall x\in\mathbb{R})[0<|x-2|<\varepsilon\Rightarrow|3x-6|<\delta]$

* this is hard! we will teach this later.

$\Leftrightarrow (\forall\varepsilon>0)(\exists\delta>0)(\exists x\in\mathbb{R})[0<|x-2|<\varepsilon\wedge|3x-6|\ge\delta]$

False. for any $\varepsilon>0$, if $\delta=\frac{\varepsilon}{6}$, choosing $x=2+\frac{\varepsilon}{3}$

$0<\underbrace{|x-2|}_{=\frac{\varepsilon}{3}}<\varepsilon$

$3x-6=6+\varepsilon-6=\varepsilon$.

$|3x-6|=\varepsilon>\delta=\frac{\varepsilon}{6}$

$|3x-6|\ge\delta$

$|3(x-2)|\ge\delta$

$3|x-2|\ge\delta$

$|x-2|\ge\frac{\delta}{3}$

(7) $\neg(\exists \varepsilon>0)(\forall \delta>0)(\forall x\in\mathbb{R})[0<|x-2|<\varepsilon \Rightarrow 0<2]$

$\Leftrightarrow (\forall \varepsilon>0)(\exists \delta>0)(\exists x\in\mathbb{R})[0<|x-2|<\varepsilon \wedge 0\geq 2]$.

True! "$P \Rightarrow$ (true statement)" is always true. (a "tautology")

* $(\forall x\in\emptyset)\ P(x)$ — always true
* $(\exists x\in\emptyset)\ P(x)$ — always false.

Problem 3

Show that the following are equivalent, no matter the truth values of P and Q, by enumerating through all four combinations of truth values of P and Q and computing the truth value on both sides of $\Leftrightarrow$:

1. $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.
2. $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$.

(1)

P	Q	$P\Rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

(2)

P	Q	$P\Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Problem 4

For each of the following, give a definition in symbols.

1. The integer n is divisible by the integer m.
2. The real number r is rational.
3. The integer d is the greatest common divisor of the integers m and n.
4. The set S has a maximum.

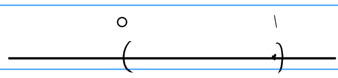
(1) $(\exists d\in\mathbb{Z})[n=dm] \leftarrow$ "$m|n$" is shorthand for

(2) $(\exists p\in\mathbb{Z})(\exists q\in\mathbb{Z})[q\neq 0 \wedge r=\frac{p}{q}]$.

(3) $d|m \wedge d|n \wedge \underbrace{(\forall x\in\mathbb{Z})[x>d \Rightarrow \neg(x|m) \vee \neg(x|n)]}_{\text{for any integer } x \text{ that is greater than } d,\ x \text{ can't divide both } m \text{ and } n}$

(4) $(\exists m\in S)(\forall x\in S)[m\geq x]$.

What about $(\exists x\in\mathbb{R})(\forall s\in S)(s<x)$.



$\forall s\in(0,1),\ s<1$

$(0,1)$ doesn't have a maximum!

Problem 5
Let A, B, C be contained in some universe of discourse U. Prove the following statements:

1. If $A \subseteq B$ then $B^C \subseteq A^C$.
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
3. $(A^C)^C = A$.
4. $A \setminus B = A \cap B^C$.

(3) We prove $(A^c)^c \subseteq A$ <u>and</u> $A \subseteq (A^c)^c$.

Let $x \in (A^c)^c$.

Then $\neg(x \in A^c)$ or $\neg(\neg\, x \in A)$

so $x \in A$. Therefore $(A^c)^c \subseteq A$.

Let $x \in A$.

Then $\neg(\neg\, x \in A)$. So $\neg(x \in A^c)$ or $x \in (A^c)^c$.

Therefore $A \subseteq (A^c)^c$.

Problem 6
Answer the following questions. Give a brief argument for your answer:

1. If A and B are finite sets, what is $|A \times B|$?
2. If A and B are countable sets, is $A \times B$ countable? If it's true in general, explain why. If it's not, come up with countable A and B so that $A \times B$ is not countable.
3. If A and B are finite sets, and $\mathcal{F}$ is the set of all functions $A \to B$, what is $|\mathcal{F}|$?
4. If $J_n = \{1, 2, 3, \ldots, n\}$ and $\mathcal{G}$ is the set of all *bijective* functions $J_n \to J_n$, what is $|\mathcal{G}|$?
5. Define $\mathcal{N} = \{S \subseteq \mathbb{N} : S \text{ is finite}\}$. In other words, $\mathcal{N}$ is the set of finite subsets of $\mathbb{N}$. Is $\mathcal{N}$ countable? Can you construct an injection from $\mathcal{N}$ into $\mathbb{N}$?

(1) $A \times B = \{(x,y) : x \in A, y \in B\}$

$\{\text{lunch}, \text{dinner}\} \times \{\text{sushi}, \text{juice}\}$

$= \{(\text{lunch}, \text{sushi}), (\text{lunch}, \text{juice}), (\text{dinner}, \text{sushi}), (\text{dinner}, \text{juice})\}$.

$|A \times B| = |A| \times |B|$.

(2) Yes, $A \times B$ is countable if A and B are countable.

$A = \{a_1, a_2, \ldots\}$

$B = \{b_1, b_2, \ldots\}$

$(a,b) = (c,d) \iff a = c$ <u>and</u> $b = d$.

1 (a_1,b_1)	2 (a_1,b_2)	4 (a_1,b_3)	...
3 (a_2,b_1)	5 (a_2,b_2)	(a_3,b_3)	
6 (a_3,b_1)	(a_3,b_2)	(a_3,b_3)	

(3) $A = \{1,2,3\}$ $B = \{1,2\}$

$f_1, f_2, \dots, f_6, f_7, f_8 : A \to B$

$f_1(1) = 1, f_1(2) = 1, f_1(3) = 1.$

$f_2(1) = 1, f_2(2) = 1, f_2(3) = 2$

$f_3(1) = 1, f_3(2) = 2, f_3(3) = 1$

$\vdots$

$f_8(1) = 2, f_8(2) = 2, f_8(3) = 2.$

$|\text{set of functions from } A \to B| = |B|^{|A|}$

(4) a <u>permutation</u> is a bijection from a set to itself.

there are $n!$ permutations of $\{1, \dots, n\}$.

"f^{-1}" only makes sense if f is invertible.

<u>Def</u> $f : A \to B$ is <u>invertible</u> if there exists a fctn $g : B \to A$ such that

$f \circ g = id_B$, $g \circ f = id_A$.

right invertible left invertible

if such g exists, we call g $\boxed{\text{the}}$ <u>inverse</u> of f, denoted f^{-1}.

i can use "the" instead of "an" because the inverse is unique, as i am about to prove:

Suppose $g, g' : B \to A$ are such that

$f \circ g = id_B = f \circ g'$, $g \circ f = id_A = g' \circ f$.

we show $g = g'$.