MAT157 TUT10

Problem 1 (Cauchy Mean Value Theorem)

1. Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Show there exists a number $x \in (a, b)$ such that
$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$
Hint: apply mean value theorem to $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$.

2. What happens if f is the identity? $(b-a)g'(x) = g(b) - g(a) \leftarrow$ MVT

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Define $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$

h cts on $[a,b]$, diff'ble on (a,b).

$$h(a) = f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a)$$
$$= f(a)g(b) - f(b)g(a)$$

$$h(b) = f(b)g(b) - f(b)g(a) - g(b)f(b) + g(b)f(a)$$
$$= f(a)g(b) - f(b)g(a)$$

$h(b) - h(a) = 0$

by MVT, $\exists\, x \in (a,b)$ s.t. $h'(x) = \dfrac{h(b) - h(a)}{b - a} = 0$

$$h'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)] = 0$$

$$f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)] \quad \square$$

Given two functions f and g differentiable in a neighbourhood around c, **L'Hôpital's Rule** states that if
$$\underbrace{\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)}_{\text{Remember to check this!}},$$
and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Problem 2

Consider the following:

$\lim_{x\to 1} 3x^2 - 6x + 4 = 1$

$$\lim_{x\to 1} \frac{x^3 - 3x^2 + 4x - 2}{2x^2 - 3x + 1} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{3x^2 - 6x + 4}{4x - 3} \cancel{\overset{\text{L'H}}{=}} \lim_{x\to 1} \frac{6x - 6}{4} = 0.$$

$\lim_{x\to 1} 4x - 3 = 1$

What is wrong? How can we fix it?

$$= \frac{3(1)^2 - 6(1) + 4}{4(1) - 3} = \frac{1}{1} = 1.$$

denom $\neq 0$

Problem 3

1. Compute $\lim_{x\to 0} \frac{x}{\tan x}$.
2. Compute $\lim_{x\to 0} \left(\cot(x) - x\csc^2(x)\right)$.[a]

[a] $\cot(x) = \frac{\cos(x)}{\sin(x)}$, $\csc(x) = \frac{1}{\sin(x)}$.

(1) 10:38.

$$\lim_{x\to 0} \frac{x}{\tan(x)}$$

$$= \lim_{x\to 0} \frac{1}{\sec^2(x)}$$

$$= \frac{1}{\sec^2(0)} = \cos^2(0) = 1.$$

$\lim_{x\to 0} x = 0$

$\lim_{x\to 0} \tan x = 0$

Check before using L'H.

(2) $\lim_{x\to 0} \left(\cot(x) - x\csc^2(x)\right)$

$$= \lim_{x\to 0} \left(\frac{\cos(x)}{\sin(x)} - \frac{x}{\sin^2(x)}\right)$$

$$= \lim_{x\to 0} \left(\frac{\cos(x)\sin(x) - x}{\sin^2(x)}\right)$$

$\lim_{x\to 0} \cos(x)\sin(x) - x = 0$

$\lim_{x\to 0} \sin^2(x) = 0$

$$\overset{L'H}{=} \lim_{x\to 0} \left(\frac{-\sin^2(x) + \cos^2(x) - 1}{2\sin(x)\cos(x)}\right)$$

$\lim_{x\to 0} -\sin^2(x) + \cos^2(x) - 1 = 0$

$\lim_{x\to 0} 2\sin(x)\cos(x) = 0$,

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-2\sin(x)\cos(x) - 2\cos(x)\sin(x)}{2\cos^2(x) - 2\sin^2(x)}.$$

$$= \frac{2\sin(0)\cos(0) - 2\cos(0)\sin(0)}{2\cos^2(0) - 2\sin^2(0)}$$

$$= \frac{0}{2} = 0.$$

Problem 4

If f and g are differentiable and $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, does it follow that $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists?

$f(x) = \sin x$

$g(x) = \cos x$

$$\lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0.$$

$$\lim_{x\to 0} \frac{\cos(x)}{-\sin(x)} \quad \text{"} \frac{1}{0} \text{"}$$

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