MAT157 TUT11

(probably start at 10:15 today, just to give you a proper 10 minute break)

Let $f : I \to \mathbb{R}$ be C^n, and $a \in I$. We define the **n-th order Taylor polynomial of f at a** as the polynomial

$$p^f_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

If the context is clear, we may write $p_{n,a}(x)$ instead of $p^f_{n,a}(x)$. We also defined the "remainder" $r^f_{n,a}(x)$ (or $r_{n,a}(x)$) of the Taylor polynomial, which is the difference between f and its n-th order Taylor polynomial approximation at a:

$$r^f_{n,a}(x) = f(x) - p^f_{n,a}(x).$$

We have proven a few facts about polynoimal approximations in class:

(i) $p_{n,a}(x)$ is a *good n-th order approximation of f at a*. That is,

$$\lim_{x\to a} \frac{r_{n,a}(x)}{(x-a)^n} \left(= \lim_{x\to a} \frac{f(x) - p_{n,a}(x)}{(x-a)^n} \right) = 0.$$

In fact, it is the *only* n-th order polynomial that is a good n-th order approximation of f at a.

(ii) $(r_{n,a})^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n$. As $r_{n,a}(x) = f(x) - p_{n,a}(x)$, this can also be written

$$(p_{n,a})^{(k)}(a) = f^{(k)}(a).$$

(iii) If f is also C^{n+1}, then for $x > a$, there exists some $c \in (a, x)$ such that

$$r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

For $x < a$, there exists some $c \in (x, a)$ such that the above equation holds.

Problem 1
Suppose p is a good n-th order approximation of f at a:

$$\lim_{x\to a} \frac{f(x) - p(x)}{(x-a)^n} = 0.$$

Show that p is a good k-th order approximation of f at a for all $k = 0, 1, \ldots, n-1$ as well.

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$$\lim_{x\to a} \frac{f(x)-p(x)}{(x-a)^k} = \lim_{x\to a} \frac{f(x)-p(x)}{(x-a)^n} \cdot (x-a)^{n-k}$$

$$= \lim_{x\to a} \frac{f(x)-p(x)}{(x-a)^n} \cdot \lim_{x\to a} (a-a)^{n-k}$$

$$= 0 \cdot 0 = 0.$$

Problem 2

1. Find the n-th order Taylor polynomial approximation of cos at $a = 0$.
2. Using fact (iii), find a large enough n so that the nth-order Taylor polynomial of cos at $a = 0$ approximates $\cos(1)$ with an error of less than 10^{-3}. That is, find an n so that
$$|r_{n,a}(1)| \leq 10^{-3}.$$
3. Calculate $\cos(1)$ correct to 3 decimal places.

1.
$$\cos(0) = 1$$
$$\cos'(0) = -\sin(0) = 0$$
$$\cos''(0) = -\cos(0) = -1$$
$$\cos^{(3)}(0) = \sin(0) = 0$$
$$\cos^{(4)}(0) = \cos(0) = 1$$

$$\cos^{(k)}(0) = \begin{cases} 1 & k = 4n \\ 0 & k = 4n+1 \\ -1 & k = 4n+2 \\ 0 & k = 4n+3 \end{cases}$$

$$p^f_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

$$= \frac{\cos(0)}{0!} + \cancel{\frac{\cos'(0)}{1!}(x-a)} + \frac{\cos''(0)}{2!}(x)^2 + \cdots + \frac{\cos^{(n)}(0)}{n!}(x)^n.$$

$$1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{\cos^{(n)}(0)}{n!}x^n.$$

2. Using fact (iii), find a large enough n so that the nth-order Taylor polynomial of cos at $a = 0$ approximates $\cos(1)$ with an error of less than 10^{-3}. That is, find an n so that

$$|r_{n,a}(1)| \leq 10^{-3}.$$

Try to do the following problem with as little aid from calculators as possible. You may find the following calculations useful:

$$2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256, 2^9 = 512, 2^{10} = 1024.$$

$$3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, 3^6 = 729, 3^7 = 2187, 3^8 = 6561.$$

$$2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40320.$$

(iii) If f is also C^{n+1}, then for $x > a$, there exists some $c \in (a, x)$ such that

$$\cos(x) - P_{n,a}(x) = r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

For $x < a$, there exists some $c \in (x, a)$ such that the above equation holds.

$$|r_{n,a}(1)| = \frac{|\cos^{(n+1)}(c)|}{(n+1)!} |1|^{n+1} \quad \text{for some } c \in (a,x).$$

$$\leq \frac{1}{(n+1)!}$$

for $n=6$

$$\frac{1}{(n+1)!} = \frac{1}{5040} < \frac{1}{1000} = 10^{-3}.$$

$$|\cos(1) - P_{6,0}(1)| < 10^{-3}$$

3. Calculate $\cos(1)$ correct to 3 decimal places.

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$$P_{6,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$P_{6,0}(1) = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$$

$$= \frac{720 - 360 + 30 - 1}{720} = \frac{389}{720} =$$

389 / 720

0.540277777778

cos(1)

0.54030230587

Let $x \in \mathbb{R}$. We defined the **open ball of radius** r **around** x, $B_r(x)$, as the set $(x - r, x + r)$. Given a set $U \subseteq \mathbb{R}$, and a point $a \in \mathbb{R}$, we say:

- a is an **interior point** of U if there exists $r > 0$ so that $B_r(a) \subseteq U$.
- a is a **boundary point** of U if for every $r > 0$, we have $B_r(a) \cap U \neq \emptyset$ and $B_r(a) \cap U^c \neq \emptyset$.

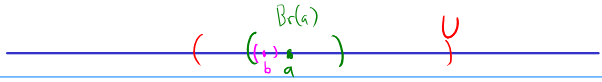
The set of interior points of U is denoted U^{int}, and the set of boundary points of U is denoted ∂U.

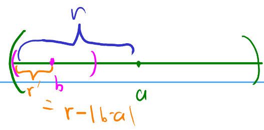
Problem 3
Find examples of sets $U \subseteq \mathbb{R}$ which:

1. Have no interior points, but have boundary points. $\{1\}$
2. Have no boundary points. $\mathbb{R}$, $\emptyset$
3. Have countably infinitely many boundary points. $\mathbb{N}$, $\mathbb{Q}$
4. Have uncountably infinitely many boundary points, and countably infinitely many interior points.

No such set exists!

Problem 4
Can a set $U \subseteq \mathbb{R}$ have finitely many interior points? No!





Prop: $U \subseteq \mathbb{R}$. Suppose $a \in U$.
Suppose $\exists r > 0$ s.t. $B_r(a) \subseteq U$.
Then $\forall b \in B_r(a)$, $b \in U^{int}$.

<u>Pf</u> Let $b \in B_r(a)$. Want $r' > 0$ s.t. 11:15
$B_{r'}(b) \subseteq B_r(a)$ $(\subseteq U)$
Choose $r' = r - |b-a| > 0$
since $b \in B_r(a)$ so $|b-a| < r$.
We show $B_{r'}(b) \subseteq B_r(a)$.
Let $c \in B_{r'}(b)$. WTS $c \in B_r(a)$ $(\Leftarrow |c-a| < r)$.

$$|c-a| \overset{T.I.}{\leq} |c-b| + |b-a|$$
$$< r' + |b-a| = r - |b-a| + |b-a|$$
$$= r.$$

So $B_{r'}(b) \subseteq B_r(a)$

We say a set $U \subseteq \mathbb{R}$ is:

- **Open** if $U^{\text{int}} = U$.
- **Closed** if $\partial U \subseteq U$.

Problem 5
Find examples of sets $U \subseteq \mathbb{R}$ which:

1. Are open and closed. $\mathbb{R}$, $\emptyset$.
2. Are open but not closed. $(0,1)$ ← $1 \in \partial(0,1)$ but $1 \notin (0,1)$
3. Are closed but not open. $[0,1]$ ← $1 \notin [0,1]^{int}$ $(= (0,1))$.
4. Are neither open nor closed. $(0,1]$. $\{\frac{1}{n} : n \in \mathbb{N}\}$.

Problem 6
Show that $U \subseteq \mathbb{R}$ is open if and only if U^c is closed.

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$\Rightarrow$ suppose $U \subseteq \mathbb{R}$ open. WTS $\partial(U^c) \subseteq U^c$
Let $x \in \partial(U^c)$. Then $\forall r > 0$, $B_r(x) \cap U^c \neq \emptyset$ and $B_r(x) \cap U \neq \emptyset$.
So $\nexists r > 0$ s.t. $B_r(x) \subseteq U$. Thus, $x \notin U^{int} \overset{U \text{ open}}{=} U$
So $x \in U^c$.

$\Leftarrow$ suppose U^c closed. WTS $U = U^{int}$ ($U^{int} \subseteq U$ we know already, so only $U \subseteq U^{int}$).
Choose $x \in U$. So $x \notin U^c$. But since U^c closed, $\partial(U^c) \subseteq U^c$, so $x \notin \partial U^c$.
Towards a contradiction that $x \notin U^{int}$,

Then for any $r>0$, $x \in B_r(x)$ so $B_r(x) \cap U \neq \emptyset$.

But since $x \notin U^{int}$, $B_r(x) \cap U^c \neq \emptyset$ as well

So $x \in \partial U = \partial(U^c) \subseteq U^c$ but we assumed $x \in U$.

$\Rightarrow \Leftarrow$

Thus $x \in U^{int}$.

Recall we have proven the following in class:

- If $\{U_i\}_{i\in I}$ is an arbitrary collection of open sets, then $\bigcup_{i\in I} U_i$ is also open.
- If $U_1, U_2, \ldots, U_n$ is a *finite* collection of open sets, then $\bigcap_{i=1}^{n} U_i$ is also open.

Problem 7

Find an infinite collection of open sets whose intersection is not open.

$U_n = (-\frac{1}{n}, \frac{1}{n}) \quad \forall n \in \mathbb{N}$.

$x \in \bigcap_{n=1}^{\infty} U_n \iff x \in U_n$ for all $n \in \mathbb{N}$.

$\bigcap_{n=1}^{\infty} U_n = \{0\} \leftarrow$ not open.

Problem 8

1. If $\{C_i\}_{i\in I}$ is an arbitrary collection of closed sets, show that $\bigcap_{i\in I} C_i$ is also closed.
2. If $C_1, C_2, \ldots, C_n$ is a *finite* collection of closed sets, show that $\bigcup_{i=1}^{n} C_i$ is also closed.
3. Show that finiteness is necessary in 2. In other words, find an infinite collection of closed sets whose union is not closed.

1. U open $\iff U^c$ closed

U closed $\iff U^c$ open.

if $\{C_i\}_{i\in I}$ is a collection of closed sets,

$\{C_i^c\}_{i\in I}$ is a collection of open sets.

$\bigcup_{i\in I} C_i^c$ is open.

$\quad ||$

De Morgan's Laws

$(A\cap B)^c = A^c \cup B^c$

$\left(\bigcap_{i\in I} C\right)^c$ is open.

$\bigcap_{i\in I} C$ is closed.

2. $C_1^c, C_2^c, \ldots, C_n^c$ are open.

$\bigcap_{i=1}^{n} C_i^c$ is open.

$\quad ||$

$\left(\bigcup_{i=1}^{n} C_i\right)^c$ is open.

$\bigcup_{i=1}^{n} C_i$ is closed.

3. $U_n = [0, 1-\frac{1}{n}] \; n \in \mathbb{N}$ $\quad \bigcup_{i=1}^{\infty} U_i = [0,1) \leftarrow$ not closed.