Recall the following summation identity:

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 1
Let $f : [0,1] \to \mathbb{R}, f(x) = x^2$, and $P = \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\}$. Compute $U(f,P)$ and $L(f,P)$. Conclude that f is integrable.

$$\begin{aligned} U(f,P) &= \sum_{i=0}^{N-1} \left(\frac{i+1}{N} - \frac{i}{N}\right) \sup_{x \in (i/N,(i+1)/N)} f(x) \\ &= \sum_{i=0}^{N-1} \frac{1}{N} \left(\frac{i+1}{N}\right)^2 \end{aligned}$$

Similarly,

$$L(f,P) = \sum_{i=0}^{N-1} \frac{1}{N} \left(\frac{i}{N}\right)^2$$

$$\begin{aligned} \Rightarrow U(f,P) - L(f,P) &= \frac{1}{N} \left(\sum_{i=0}^{N-1} \left(\frac{i+1}{N}\right)^2 - \left(\frac{i}{N}\right)^2\right) \\ &= \frac{1}{N} \left(\left(\frac{N-1+1}{N}\right)^2 - \left(\frac{0}{N}\right)^2\right) \\ &= \frac{1}{N} \end{aligned}$$

■

Given $\varepsilon > 0$, we can pick N sufficiently large, construct the partition depending on N as above, so that $1/N < \varepsilon$, for which $U(f,P) - L(f,P) = 1/N < \varepsilon$ by the work above.

Problem 2
Let $f : [a,b] \to \mathbb{R}$ be an integrable function and suppose that $[c,d] \subseteq [a,b]$. Show that the restriction of f to $[c,d]$ is also integrable.

Define the function $g : [c,d] \to \mathbb{R}$, where $g(x) = f(x)$ for all $x \in [c,d]$. We want to show that g is integrable.

Let $\varepsilon > 0$ be given, we can find a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Assume without the loss of generality that P contains c and d, since adding them in is no problem if we need to.

Then we can write $P = \{x_1, \ldots x_i, c, \ldots d, x_j, \ldots x_k\}$.

Let $P_1 = \{x_1 \ldots x_i\}$, let $P_2 = \{c, \ldots d\}$, and let $P_3 = \{x_j \ldots x_k\}$. Here, P_2 is a partition of $[c,d]$.

For simplicity of notation let

$$u(f,n) = (x_{n+1} - x_n) \sup_{x \in (x_n, x_{n+1})} f(x)$$

and let

$$l(f,n) = (x_{n+1} - x_n) \inf_{x \in (x_n, x_{n+1})} f(x)$$

Then

$$U(f,P) = \sum_{n=1}^{i} u(f,n) + U(g,P_2) + \sum_{n=j}^{k} u(f,n)$$

$$L(f,P) = \sum_{n=1}^{i} l(f,n) + L(g,P_2) + \sum_{n=j}^{k} l(f,n)$$

Then

$$U(f,P) - L(f,P) = U(g,P_2) - L(g,P_2) + \sum_{n=1}^{i}(u(f,n) - l(f,n)) + \sum_{n=j}^{k}(u(f,n) - l(f,n))$$

Therefore, $\varepsilon > U(f,P) - L(f,P) \geq U(g,P_2) - L(g,P_2)$ ■

Problem 3
Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that f is integrable. Generalize this result to $[a,b]$ instead of $[0,1]$. *Hint: Recall that continuous over a closed interval implies uniformly continuous.*

Idea: Since $U(f,P) - L(f,P) = \sum_{i=1}^{n}(u(f,i) - l(f,i))$, we can rewrite this:

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(x_{i+1} - x_i)(\sup_{x\in(x_i,x_{i+1})} f(x) - \inf_{x\in(x_i,x_{i+1})} f(x))$$

$$= \sum_{i=1}^{n}(x_{i+1} - x_i)(\sup_{x,y\in(x_i,x_{i+1})} (f(x) - f(y))$$

Since f is uniformly continuous, we can bound $f(x) - f(y)$ globally.
Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow f(x) - f(y) < |f(x) - f(y)| < \varepsilon$$

$$\Rightarrow (\sup_{x,y\in(x_i,x_{i+1})} (f(x) - f(y)) < \varepsilon$$

So we can reach the above condition simply by taking a partition P with $l(P) < \delta$.

Problem 4

1. Let $f : [a,b] \to \mathbb{R}$ be continuous, except at one point $c \in [a,b]$. Also suppose f is bounded. Show that f is still integrable.

2. Let $f : [a,b] \to \mathbb{R}$ be continuous, except at finitely many points $c_1, \ldots, c_n \in [a,b]$. Also suppose f is bounded. Using induction, show that f is still integrable.

1) For $\delta > 0$, write

$$[a,b] = [a, c-\delta] \cup [c-\delta, c+\delta] \cup [c+\delta, b]$$

We know that f is continuous on the first and third intervals, therefore it's integrable on them.
For the middle interval, Consider the trivial partition $\{c-\delta, c+\delta\}$. Then

$$U(f,P) - L(f,P) = 2\delta(\sup_{x\in(c-\delta,c+\delta)} f(x) - \inf_{x\in(c-\delta,c+\delta)} f(x))$$

Let M be the global bound of f, such that $|f(x)| < M$ for every $x \in [a,b]$.
Then, clearly,

$$2\delta(\sup_{x\in(c-\delta,c+\delta)} f(x) - \inf_{x\in(c-\delta,c+\delta)} f(x)) < 2\delta M$$

Formally, let $\varepsilon > 0$ be given. Choose δ small enough so that $\delta < \varepsilon/6M$. Choose partitions P_1, P_2 of $[a, c-\delta], [c+\delta, b]$ such that the difference of sums on P_1 and P_2 are both less than $\varepsilon/3$. Then $P_1 \cup P_2$ forms a partition of P such that

$$\begin{aligned} U(f,P) - L(f,P) &< \varepsilon/3 + 2\delta M + \varepsilon/3 \\ &< \varepsilon/3 + 2\frac{\varepsilon}{6M}M + \varepsilon/3 \\ &= \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

2) The base case is part 1). Assume we know f being integrable and bounded except on k points implies that f is integrable. Add in another point, then we can apply our same argument from part a). The only difference being we argue integrability from our inductive hypothesis, not by continuity.