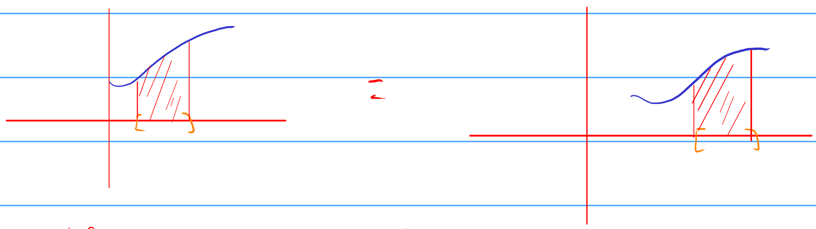
Tutorial 13
(i am still kinda sick lol, today is gonna be short hopefully)

**Problem 1**
Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Show that for any $c \in \mathbb{R}$,

$$\int_{a+c}^{b+c} f(x-c)\,dx$$

is defined (the underlying function is integrable), and

$$\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(x)\,dx.$$



10:15

Define $g(x) = f(x-c)$.

Show $g$ is integrable on $[a+c, b+c]$

WTS $\forall \varepsilon > 0$ $\exists P'$ of $[a+c, b+c]$

$U(g,P') - L(g,P') < \varepsilon$.

Let $\varepsilon > 0$. Take $P$ partition of $[a,b]$ s.t.

$U(f,P) - L(f,P) < \varepsilon$.

Write $P = \{x_0, x_1, \dots, x_n\}$.

Define $P' = \{x_0+c, x_1+c, \dots, x_n+c\}$.

$$U(g,P') = \sum_{i=1}^{n} ((x_i+c)-(x_{i-1}+c)) \cdot \sup_{x\in[x_{i-1}+c,\, x_i+c]} f(x-c) = \sup_{x\in[x_{i-1}, x_i]} f(x)$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x\in[x_{i-1}, x_i]} f(x) = U(f,P)$$

$L(g,P') = L(f,P)$

$U(g,P') - L(g,P') = U(f,P) - L(f,P) < \varepsilon$

To show integrals are equal:

$U(g,P') = U(f,P)$ $\forall P$

so $\inf_{P'} \{U(g,P')\} = \inf_{P} \{U(f,P)\} \overset{\text{since } f \text{ integrable}}{=} \int f$.

$\int g$

**Problem 2**
Suppose $f : [a, b] \to \mathbb{R}$ is bounded and monotone increasing. Show $f$ is integrable.

Lecture proved this !!

**Problem 3**

1. Show that $f : [1, a] \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is integrable, for any $a > 1$.
2. Show that for any $a > 1$ and $b > 0$,

$$\int_1^a \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt.$$

*Hint: Scale an arbitrary partition* $[x_0, x_1, \ldots, x_n]$ *of* $[1, a]$ *into the partition* $[bx_0, bx_1, \ldots, bx_n]$ *of* $[b, ab]$.

1. $\frac{1}{x}$ is bounded and decreasing on $[1, a]$ so by Problem 2 it's integrable.

2. $U(\frac{1}{x}, \underset{[1,a]}{P}) = U(\frac{1}{x}, \underset{[b,ab]}{P'})$ 10:32

$P = \{ \underset{1}{x_0}, x_1, \ldots, \underset{a}{x_n} \}$

Let $P' = \{ \underset{b}{bx_0}, bx_1, \ldots, \underset{ab}{bx_n} \}$

$$U(\tfrac{1}{x}, P') = \sum_{i=1}^{n} (bx_i - bx_{i-1}) \cdot \sup_{x \in [bx_i, bx_{i-1}]} \left(\tfrac{1}{x}\right)$$

$$= \sum_{i=1}^{n} b(x_i - x_{i-1}) \cdot \frac{1}{bx_1}$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \frac{1}{x_i}$$

$$U(\tfrac{1}{x}, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot \frac{1}{x_1}$$

Same with $L(\frac{1}{x}, P) = L(\frac{1}{x}, P')$

Using ideas from Q1, we can show $\frac{1}{x}$ integrable over $[b, ab]$ and $\int_1^a \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt$.

$$\int_b^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt$$

$$\int_1^{ab} \frac{1}{t}\,dt - \int_1^b \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt$$

$$\int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$$

$$\ln(ab) = \ln(a) + \ln(b)$$

**Problem 5**

bounded

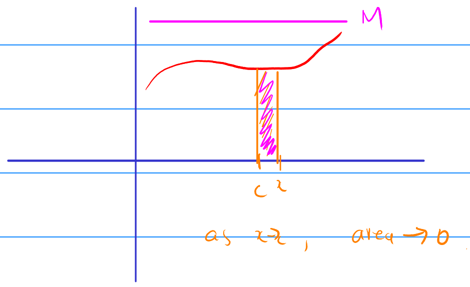
Suppose $f$ is integrable on $[a, b]$. Show that the function $F : [a, b] \to \mathbb{R}, x \mapsto \int_a^x f$ is continuous.

$\lim_{x \to c} F(x) = F(c)$ 10:50

$\Leftrightarrow \lim_{x \to c} \int_a^x f = \int_a^c f$

$\Leftrightarrow \lim_{x \to c} \left( \int_a^x f - \int_a^c f \right) = 0$

$\Leftrightarrow \lim_{x \to c} \int_c^x f = 0$




as $x \to c$, area $\to 0$

Say $|f(x)| < M \quad \forall x.$

Let $\varepsilon > 0$. Want $\delta > 0$ s.t. $|x - c| < \delta \Rightarrow \left| \int_c^x f \right| < \varepsilon$

Choose $\delta = \frac{\varepsilon}{2M}$. Then, if $|x-c| < \delta$:

Case 1: $x > c$

$\left| \int_c^x f \right| \le \int_c^x |f| \le \int_c^x M = (x-c) M \le \frac{\varepsilon}{2M} \cdot M = \frac{\varepsilon}{2} < \varepsilon$

Case 2: $x < c$

$\left| \int_c^x f \right| = \left| \int_x^c f \right| \le \int_x^c |f| \le (c-x) M < \varepsilon$

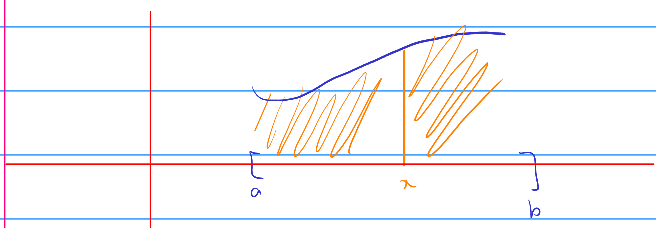
Case 3: $x = c$ $\quad \int_c^x f = 0 < \varepsilon$

**Problem 4**

Suppose $f$ is integrable on $[a, b]$. Prove that there is a number $x$ in $[a, b]$ such that

$$\int_a^x f = \int_x^b f.$$

Show by example that it is not always possible to choose $x \in (a, b)$ (i.e. $x$ might have to be on the boundary).




Define $F(x) = \int_a^x f - \int_x^b f.$

cts fctns of $x$, by problem 5. so $F$ cts.

Case 1: $\int_a^b f > 0$. $\quad F(a) = \int_a^a f - \int_a^b f = 0 - \int_a^b f < 0$

$F(b) = \int_a^b f - \int_b^b f = \int_a^b f - 0 > 0.$

Since $F$ cts, by IVT, $\exists\, c \in (a,b)$ s.t. $F(c) = 0$.

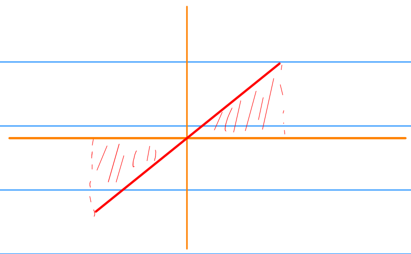
Case 2: $\int_a^b f < 0$ $\qquad F(a) = 0 - \int_a^b f > 0$

$F(b) = \int_a^b f - 0 < 0$

by IVT, $\exists\, c \in (a,b)$ s.t. $F(c) = 0$.

Case 3: $\int_a^b f = 0$ $\qquad F(a) = 0 - 0 = 0$.

Example where it's not possible to chose $x$ inside $(a,b)$:



$f(x) = x$ on $[-1, 1]$

$$\int_{-1}^{1} f = 0$$

$$\int_{-1}^{x} f = \frac{1}{2}x^2 - \frac{1}{2} \neq 0 \text{ unless } x = \pm 1.$$