Tutorial... 14? I believe. I lost count :(

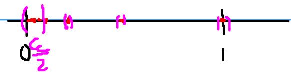
Problem 5
Let $L : (0,\infty) \to \mathbb{R}$ be defined by

$$L(x) = \int_1^x \frac{1}{t} dt.$$

1. Using a similar argument to Problem 3, i.e not using FTC, show that L is an anti-derivative of $\frac{1}{x}$ on the domain $(0,\infty)$. In this case, L is the unique anti-derivative of $\frac{1}{x}$ satisfying $L(1) = 0$.
2. Show that $L(xy) = L(x) + L(y)$ for $x, y \in \mathbb{R}$. *Hint: treat y as a constant, differentiate with respect to x, or the other way around.*
3. Show that $L(1/x) = -L(x)$ for any $x \in \mathbb{R}$, again using differentiation.
4. Show that $L(x^n) = nL(x)$ for any integer $n \neq -1$ and $x \in \mathbb{R}$.
5. Show that L is strictly increasing. Combining this with the fact that it's differentiable with continuous derivative, we can apply IFT to obtain a C^1 inverse $\exp : \mathbb{R} \to (0,\infty)$. Do you recognize this function and its inverse?

Here's Problem 5. Unfortunately it looks like Tyler covered all of this in lecture...

Problem 1
Let $S = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$. Prove that S has Jordan measure 0.



Let $\varepsilon > 0$.

$P = \left(0, \frac{\varepsilon}{2}\right) \cup \left(\frac{1}{2^0} - \frac{\varepsilon}{4n}, \frac{1}{2^0} + \frac{\varepsilon}{4n}\right) \cup \left(\frac{1}{2^1} - \frac{\varepsilon}{4n}, \frac{1}{2^1} + \frac{\varepsilon}{4n}\right)$

$\cup \dots \cup \left(\frac{1}{2^{n-1}} - \frac{\varepsilon}{4n}, \frac{1}{2^{n-1}} + \frac{\varepsilon}{4n}\right)$ covers S

$|P| \leq \frac{\varepsilon}{2} + \underbrace{\frac{\varepsilon}{2n} + \dots + \frac{\varepsilon}{2n}}_{n \text{ times}} = \varepsilon.$

combine intervals that aren't disjoint

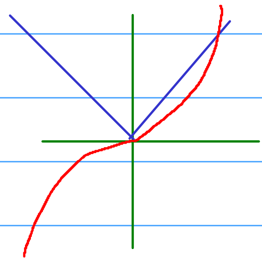
Problem 2
Show that an unbounded set cannot have Jordan measure 0.

Problem 3

For each of the following functions, find an anti-derivative. The anti-derivative must be defined wherever the function is defined.

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. $g(x) = \frac{x^4}{4}$ $\qquad g'(x) = x^3$
2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x - \cos x$ $\qquad -\cos(x) - \sin(x)$.
3. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x^2}$ $\qquad (-x^{-1})' = x^{-2}$
4. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$
5. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, where $n \in \mathbb{Z}$ and $n \neq -1$. (What goes wrong when $n = -1$? We'll explore this in Problem ~~4~~ 5) $\qquad \frac{1}{n+1} x^{n+1}$

4. $$g(x) = \begin{cases} \frac{x^2}{2} & x \geq 0 \\ -\frac{x^2}{2} & x < 0 \end{cases}$$



$$g'(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases} = |x|$$

$$g'(0) = \lim_{x \to 0} \frac{g(x)}{x} = 0$$

$$\lim_{x \to 0^+} \frac{g(x)}{x} = \lim_{x \to 0^+} \frac{x}{2} = 0$$

$$\lim_{x \to 0^-} \frac{g(x)}{x} = \lim_{x \to 0^-} \frac{-x}{2} = 0$$

Problem 4

Let $f : [0, \infty) \to \mathbb{R}$ be the function $f(x) = x^3$. Let $F : [0, \infty) \to \mathbb{R}$ be the function:

$$F(x) = \int_0^x t^3 dt.$$

1. Assume $h \in \mathbb{R}$. Without using FTC and instead using algebra and other things we know about integration, simplify the expression:

$$\frac{F(x+h) - F(x)}{h}$$

into a single real number times a single integral.

2. Show that:

$$F(x) \quad \lim_{h \to 0} \left(\frac{F(x+h) - F(x)}{h} \right) = f(x) \qquad 10{:}50$$

and use this to conclude that F is an anti-derivative of f. *Hint: you don't need anything fancy here, use the fact that f is monotone increasing.*

3. If G is any other anti-derivative of f, show that $F(x) = G(x) - G(0)$.

4. Using (3) and the anti-derivative found in Problem 2, find

$$\int_{2021}^{2022} t^3 dt.$$

Simplify your answer so that it contains no integral sign.

1. $\dfrac{\int_0^{x+h} t^3 dt - \int_0^x t^3 dt}{h} = \dfrac{\int_x^{x+h} t^3 dt}{h}$

2. $\lim_{h \to 0^+} \dfrac{\int_x^{x+h} t^3 dt}{h} \leq \lim_{h \to 0^+} \dfrac{\int_x^{x+h} (x+h)^3 dt}{h}$

$= \lim_{h \to 0^+} \dfrac{h \cdot (x+h)^3}{h}$

$= \lim_{h \to 0} (x+h)^3 = x^3 = f(x).$

$\lim_{h \to 0^+} \dfrac{\int_x^{x+h} t^3 dt}{h} \geq \lim_{h \to 0^+} \dfrac{\int_x^{x+h} x^3 dt}{h}$

$= \lim_{h \to 0^+} \dfrac{h x^3}{h} = f(x).$

by squeeze, $\lim_{h \to 0^+} \dfrac{F(x+h) - F(x)}{h} = f(x).$

Same with $h \to 0^-$.

3. If G is any other anti-derivative of f, show that $F(x) = G(x) - G(0)$.

3. [illegible]

[illegible]

4. Using (3) and the anti-derivative found in Problem 2, find

$$\int_{2021}^{2022} t^3 dt.$$

Simplify your answer so that it contains no integral sign.

[illegible] 11:03.

Let $G(x) = \frac{x^4}{4}$. $G'(x) = f(x)$.

by (3), $F(x) = G(x) - G(0) = \frac{x^4}{4}$.

$$\int_{2021}^{2022} t^3 dt = \int_0^{2022} t^3 dt - \int_0^{2021} t^3 dt$$

$$= F(2022) - F(2021)$$

$$= \frac{2022^4 - 2021^4}{4}.$$