Tutorial 15

Suppose $f : [a,b] \to \mathbb{R}$ is integrable. The **Fundamental Theorem of Calculus** states the following:

1. The function $F : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt$$

   is continuous. F is also differentiable wherever f is continuous, with $F'(x) = f(x)$ in this case.

2. Suppose that F is a continuous anti-derivative of f which is differentiable at all but finitely many points. Then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Problem 1

Let $F : (0,\infty) \to \mathbb{R}$ be defined as

$$F(x) = \int_x^{x^2} \frac{1}{t}dt$$

1. Determine for which x we have $F(x) \geq 0$ and for which x we have $F(x) < 0$.
2. Find an expression for $F'(x)$ which involves no integral signs.

1. $F(x) \geq 0$ whenever $x^2 \geq x$ $\uparrow$ $x \geq 1$ $\qquad$ $F(x) < 0$ when $x^2 < x$, $x < 1$

2. $F(x) = \int_1^{x^2} \frac{1}{t}dt - \int_1^x \frac{1}{t}dt$

   Let $g(x) = \int_1^x \frac{1}{t}dt$

   $F(x) = g(x^2) - g(x)$ $\qquad$ $g'(x) \overset{FTC}{=} \frac{1}{x}$

   $F'(x) = g'(x^2)\cdot 2x - g'(x)$.

   $= \frac{1}{x^2}\cdot 2x - \frac{1}{x} = \frac{2}{x} - \frac{1}{x} = \frac{1}{x}$.

Problem 2

1. Prove that if f is integrable on $[a,b]$ and $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

$$\int_a^b f(x)\ dx = (b-a)\mu$$

   for some number μ with $m \leq \mu \leq M$.

Since $m \leq f(x) \leq M$

$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx$

$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

$m \leq \underbrace{\frac{\int_a^b f(x)\,dx}{b-a}}_{\text{Let } \mu =} \leq M$

2. Prove that if f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

for some $c \in [a,b]$. Give a counterexample to this when f is not continuous on $[a,b]$.

Let $m = \min\{f(x) : x \in [a,b]\}$ (exists by EVT)

$M = \max\{f(x) : x \in [a,b]\}$

by part 1, there is some $m \le \mu \le M$

$\int_a^b f(x)\,dx = (b-a)\mu$ (want to find $c \in [a,b]$ s.t. $f(c) = \mu$)

Since f cts, $\exists\, x^{min} \in [a,b]$ s.t. $f(x^{min}) = m$

$x^{max} \in [a,b]$ $f(x^{max}) = M$.

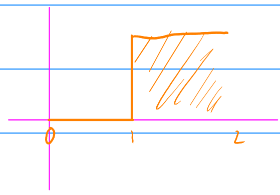
$\Rightarrow$ by IVT, $\exists\, c$ between x^{min} and x^{max} s.t.

$f(c) = \mu$.

Counterexample: $f(x) = \begin{cases} 0 & x \in [0,1] \\ 1 & x \in [1,2] \end{cases}$

$\int_0^2 f(x) = 1 = \frac{1}{2}(b-a)$

but $f(x) \ne \frac{1}{2}$ anywhere.



3. Let f be continuous on $[a,b]$ and g be integrable and nonnegative on $[a,b]$. Prove that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

for some $c \in [a,b]$. You may assume that the integral exists.

Let $m = \min f(x)$

$M = \max f(x)$

$m \int_a^b g(x)dx \le \int_a^b f(x)g(x)\,dx \le M \int_a^b g(x)\,dx$

$m \le \underbrace{\dfrac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}}_{\text{Let } \mu =} \le M$ (if $\int_a^b g(x)\,dx \ne 0$)

Since f cts, by IVT, $\exists\, c \in [a,b]$ s.t. $f(c) = \dfrac{\int_b^a f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$

So $f(c) \int_a^b g(x)\,dx = \int_a^b f(x) g(x)\,dx$

if $\int_a^b g(x)\,dx = 0$, then $m \int_a^b g(x)\,dx = 0$, $M \int_a^b g(x)\,dx = 0$

Since

$$0 = m \int_a^b g(x)\,dx = \int_a^b f(x) g(x)\,dx \le M \int_a^b g(x)\,dx = 0,$$

So $\int_a^b f(x) g(x)\,dx = 0$

Choose any c we want, we get

$$\int_a^b f(x) g(x)\,dx = f(c) \int_a^b g(x)\,dx.$$

Problem 3
Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Show that there does not exist a function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$. *Hint: Recall that if $F'(x) = 0$ along an interval, then F is constant on that interval.*

Also, find $\int f$. This shows that we can't always use the FTC to find the integral, and that defining the integral as the "inverse" of the derivative is too restrictive.

$\int f = 0.$

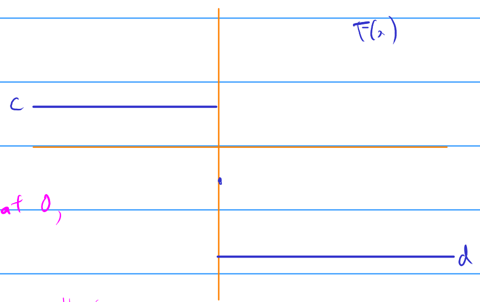
Suppose $F'(x) = f(x)$ towards contradiction.

On the interval $(-\infty, 0)$, $F'(x) = 0$.

So $F(x)$ constant along $(-\infty, 0)$, say $F(x) = c$ for $x \in (-\infty, 0)$.

Same with $(0, \infty)$

$F(x) = d$ for $x \in (0, \infty)$.



$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since F diffable at 0,

F cts at 0.

Then it must happen that $c = d = F(0)$.

(or else F not cts at 0) So F is constant.

$\Rightarrow F'(x) = 0$ everywhere

$\Rightarrow F'(0) \neq 1 = f(0)$

contradiction as $F' = f$.

Problem 4
For a fixed $a \in \mathbb{R}$, consider the function $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) = \int_0^a \left(\int_a^x st^2 dt \right) ds$$

What is F'? *Hint: Move outside.*

$F(x) = \int_0^a \left(\int_a^x st^2\,dt \right) ds$

$= \int_0^a \left(s \int_a^x t^2\,dt \right) ds$ since s is constant w.r.t t

$= \left(\int_a^x t^2\,dt \right) \int_0^a s\,ds$ $\quad \int_a^x t^2\,dt$ is constant w.r.t. s

$\overset{FTC}{=} \left(\frac{1}{2} a^2 - 0 \right) \cdot \int_a^x t^2\,dt$

$= \frac{a^2}{2} \int_a^x t^2\,dt$

$F'(x) = \frac{a^2}{2} x^2$ by FTC.

Recall the integration by parts formula:

$$\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\,dx,$$

or the mnemonic,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

$\int u\,dv = uv - \int v\,du$

Problem 5
Let $I = \int e^x \sin(x)dx$. Apply integration by parts (twice) to obtain an expression for I in terms of itself. Solve for I to obtain an expression with no integral sign.

$\int e^x \sin(x)\,dx$ $\quad u = \sin x \quad du = \cos(x)\,dx$
$\quad dv = e^x\,dx \quad v = e^x$

$= e^x \sin x - \int e^x \cos x\,dx$ $\quad u = \cos x \quad du = -\sin x\,dx$
$\quad dv = e^x dx \quad v = e^x$

$= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x)\,dx \right)$

$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$2 \int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x\,dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$

Check: $\frac{d}{dx}\left(\frac{1}{2}(e^x \sin x - e^x \cos x) \right)$

$= \frac{1}{2}\left(e^x \sin x + \cancel{e^x \cos x} - (\cancel{e^x \cos x} - e^x \sin x) \right)$

$= \frac{1}{2}(2 e^x \sin x) = e^x \sin x.$