Tutorial 16...? i believe

You'll need these two integration techniques:

- Integration by substitution (setting $u = g(x)$):

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

  For indefinite integrals,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

  where you find an antiderivative of $f(u)$ on the right side, and then substitute $u = g(x)$ to get back the answer in terms of x.

- Integration by parts (setting $u = f(x), v = g'(x)$):

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b g(x)f'(x)\,dx.$$

  By "abuse of notation", we write it

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

  For indefinite integrals, it becomes

$$\int u\,dv = uv - \int v du.$$

Problem 1
Compute the following indefinite integrals.

1. $\int \sin^2(x)\cos^3(x)\,dx$. *Hint:* $\cos^2(x) = 1 - \sin^2(x)$.
2. $\int \sin(x)\cos^3(x)\,dx$.
3. $\int \cos^2(x)\,dx$. *Hint: There is an easy way, and there is a hard way.*
4. $\int \sin^2(x)\cos^2(x)\,dx$.
5. $\int \log(x)\,dx$.
6. $\int x^n e^x\,dx$, for $n \in \mathbb{N}$.

1. $\int \sin^2(x) \cos^3(x)\, dx$

$= \int \sin^2(x) \cos^2(x) \cos(x)\, dx$

$= \int \sin^2(x)(1 - \sin^2(x)) \cos(x)\, dx$ $\qquad u = \sin(x)$, $du = \cos(x)\, dx$

$= \int u^2 (1 - u^2)\, du$

$= \int u^2 - u^4\, du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 du + C$

$= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$

2. $\int \sin(x) \cos^3(x)\, dx$ $\qquad u = \cos(x)$ $\qquad du = -\sin(x)\, dx$

$-\int u^3\, du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4(x) + C$

3. $\int \cos^2(x)\, dx$

Method 1:

$\int \cos^2(x)\, dx$ $u = \cos(x)$ $du = -\sin(x)\,dx$ $dv = \cos(x)\,dx$ $v = \sin(x)$

$= \sin(x)\cos(x) + \int \sin^2(x)\,dx$ $u = \sin(x)$ $du = \cos(x)\,dx$ $dv = \sin(x)\,dx$ $v = -\cos(x)$

$= \sin(x)\cos(x) - \sin(x)\cos(x) + \int \cos^2(x)\,dx$

$= \int \cos^2(x)\,dx$ X

Method 2:

$\cos(2x) = \cos^2(x) - \sin^2(x)$
$= 2\cos^2(x) - 1$

$\cos^2(x) = \frac{1}{2}(\cos(2x)+1)$

$\int \cos^2(x)\,dx = \frac{1}{2}\int \cos(2x)+1\,dx$
$= \frac{1}{2}\left(\frac{1}{2}\sin(2x) + x + C\right)$
$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + C$

$\int \sin^2(x)\cos^2(x)\,dx$

$= \int (1-\cos^2(x))\cos^2(x)\,dx$

$= \int \cos^2(x)\,dx - \int \cos^4(x)\,dx$

$= \frac{1}{4}\sin(2x) + \frac{1}{2}x - \frac{1}{32}\sin(4x) - \frac{3}{8}x - \frac{1}{4}\sin(2x) + C$

$= -\frac{1}{32}\sin(4x) + \frac{1}{8}x + C$

$\int \cos^4(x)\,dx$

$= \int (\cos^2(x))^2\,dx$

$= \int \left(\frac{1}{2}(\cos(2x)+1)\right)^2$

$= \int \frac{1}{4}(\cos^2(2x) + 2\cos(2x) + 1)\,dx$

$= \frac{1}{4}\int \cos^2(2x)\,dx + \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int 1\,dx$

$= \frac{1}{32}\sin(4x) + \frac{1}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{4}x + C$

$\int \cos^2(2x)\,dx$
$u = 2x$
$du = 2dx$
$= \frac{1}{2}\int \cos^2(u)\,du$
$= \frac{1}{2}\left(\frac{1}{4}\sin(2u) + \frac{1}{2}u + C\right)$
$= \frac{1}{8}\sin(4x) + \frac{1}{2}x + C$

example. $\int \sin^2(x)\cos^2(x)\,dx$

$$= \int (\sin(x)\cos(x))^2\,dx = \int \left(\frac{\sin(2x)}{2}\right)^2 dx \qquad u = 2x$$
$$\qquad du = 2dx$$
$$= \frac{1}{4}\int \sin^2(2x)\,dx$$
$$= \frac{1}{4}\int \frac{\sin^2(u)}{2}\,du$$
$$= \frac{1}{8}\int \frac{1-\cos(2u)}{2}\,du$$

half angle formula $\qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$

$$= \frac{1}{16}\int (1-\cos(2u))\,du$$
$$= \frac{1}{16}\left(u - \frac{\sin(2u)}{2}\right) du$$
$$= \frac{1}{16}\left(2x - \frac{\sin(4x)}{2}\right) + C = \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

5. $\int \log x \; dx \qquad u = \log x \quad du = \frac{1}{x}\,dx$

$\qquad dv = 1\,dx \quad v = x$

$$= x\log x - \int 1\,dx$$
$$= x\log x - x + C$$

6. $\int x^n e^x\,dx \qquad u = x^n \quad du = n x^{n-1}\,dx$

$\qquad dv = e^x dx \quad v = e^x$

$$= x^n e^x - n\int x^{n-1} e^x\,dx \qquad u = x^{n-1} \quad du = (n-1)x^{n-2}\,dx$$
$$\qquad dv = e^x dx \quad v = e^x$$
$$= x^n e^x - n\left(x^{n-1}e^x - (n-1)\int x^{n-2} e^x\,dx\right)$$
$$= x^n e^x - n x^{n-1} e^x - n(n-1)\int x^{n-2} e^x\,dx$$
$$= x^n e^x - n x^{n-1} e^x + n(n-1)x^{n-2}e^x - n(n-1)(n-2)\int x^{n-3} e^x\,dx$$
$$= x^n e^x - n x^{n-1} e^x + n(n-1)x^{n-2}e^x - \cdots \quad n!\,e^x + C$$
$$= \left(\sum_{i=0}^{n} \frac{n!}{(n-i)!}(-1)^i x^{n-i} e^x\right) + C$$

Problem 2
For which $\alpha \in \mathbb{R}$ does the improper integral $\int_0^1 x^\alpha \, dx$ converge? What about $\int_1^\infty x^\alpha \, dx$?

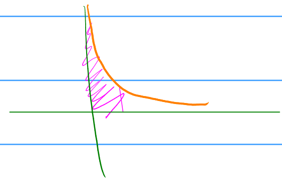
p-test

$\int_0^1 x^\alpha dx$

$= \lim_{c \to 0^+} \int_c^1 x^\alpha dx$

(if $\alpha \neq -1$) $= \lim_{c \to 0^+} \left[\frac{1}{\alpha+1} x^{\alpha+1} \right]_{x=c}^{x=1}$

$= \lim_{c \to 0^+} \frac{1}{\alpha+1} - \frac{c^{\alpha+1}}{\alpha+1}$



if $\alpha < -1$, $c^{\alpha+1} \to \infty$ as $c \to 0^+$

So lim DNE

if $\alpha > -1$, $c^{\alpha+1} \to 0$

so lim exists.

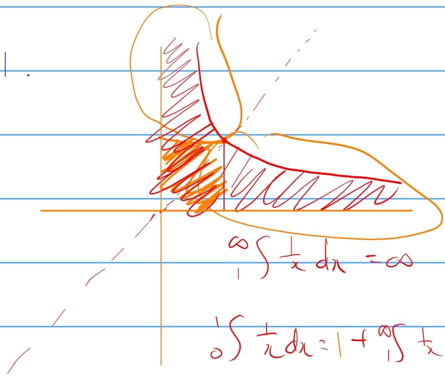
if $\alpha = -1$ (left formula won't work)

$\lim_{c \to 0^+} \int_c^1 \frac{1}{x} dx$

$= \lim_{c \to 0^+} \left(\log(1) - \log(c) \right)$

$= \infty$ (since $\lim_{c \to 0^+} \log(c) = -\infty$)

So $\int_0^1 x^\alpha dx$ converges iff $\alpha > -1$.



$\int_1^\infty \frac{1}{x} dx = \infty$

$\int_0^1 \frac{1}{x} dx = 1 + \int_1^\infty \frac{1}{x} dx$

continuous

Problem 3
Suppose $f : [1, \infty) \to \mathbb{R}$ is decreasing, nonnegative, and $\int_1^\infty f(x)\, dx$ converges.

1. Show that for every $y \in (0, f(1)]$, there is a unique x such that $f(x) = y$.
2. Define $f^{-1} : (0, f(1)] \to \mathbb{R}$. Show that

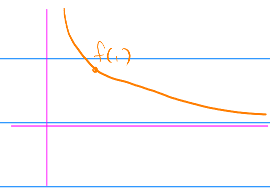
$$\int_0^{f(1)} f^{-1}(y)\, dy = \int_1^\infty f(x)\, dx.$$

Hint: Draw graphs for both and compare the area under each curve.

1. $\int_1^\infty f(x) dx$ converges, $\Rightarrow \lim_{x \to \infty} f(x) = 0$

f cts

apply IVT.



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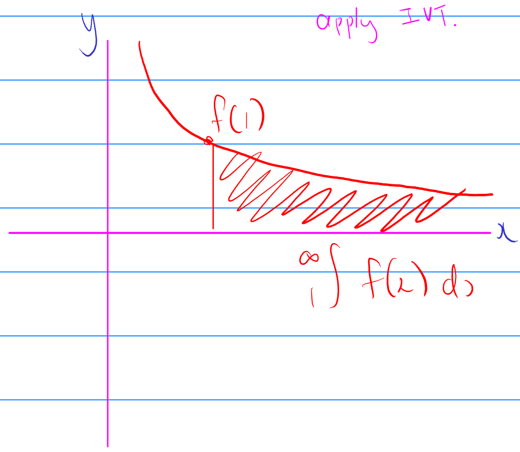
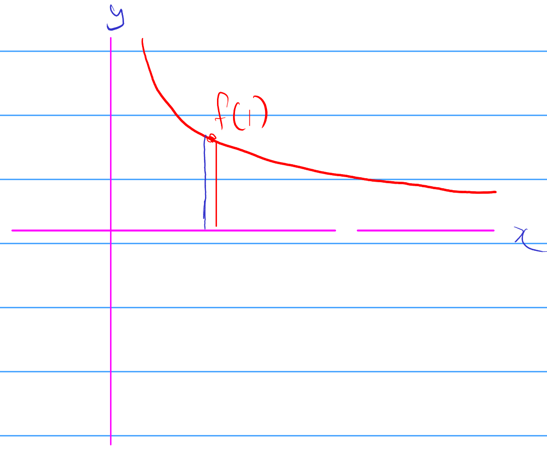


$\int_1^\infty f(x) dx$



Problem 4

1. Let $f : \mathbb{R} \to \mathbb{R}$ be an even, continuous function. Show that for any $a \in \mathbb{R}$ we have

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd, continuous function. Show that for any $a \in \mathbb{R}$ we have

$$\int_{-a}^{a} f(x)dx = 0$$

Remark: this is true for integrable functions in general, but a fun exercise in u-substitution.

Remark 2: It is tempting to look at the result of (2) and conclude that $\int_{-\infty}^{\infty} f(x)dx = 0$, but this is not the way we've defined the above integral; due to how chaotic things can get at infinity it's important the two infinities are considered separately.

1. $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$u = -x$
$du = -dx$

$$= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx.$$

2.