Tutorial 17

**Problem 1**
If $f'$ is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, show that

$$\int_0^\infty f'(x)\, dx = -f(0).$$

$$\int_0^\infty f'(x)\,dx = \lim_{b\to\infty} \int_0^b f'(x)\,dx$$

$$\overset{FTC}{=} \lim_{b\to\infty} f(x)\Big|_{x=0}^{x=b}$$

$$= \lim_{b\to\infty} \left(f(b) - f(0)\right) = -f(0)$$

**Problem 2**
State the following convergence tests.

1. $p$-test.
2. Basic comparison test.
3. Limit comparison test.
4. Absolute convergence test.

p-test : $\int_1^\infty \frac{1}{x^p}\,dx$ converges $\Leftrightarrow$ $p > 1$

$\int_0^1 \frac{1}{x^p}\,dx$ converges $\Leftrightarrow$ $p < 1$

BCT: Let $f, g : [a,\infty) \to \mathbb{R}$. If $f \geq g$ and $\int_a^\infty f$ converges, then $\int_a^\infty g$ converges.

(Contrapositive: if $\int_a^\infty g$ div. then $\int_a^\infty f$ div.)

LCT: Let $f, g : [a, \infty) \to \mathbb{R}$, with $f, g > 0$.
Suppose $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and $\neq 0$.
Then $\int_a^\infty f$ conv. $\Leftrightarrow$ $\int_a^\infty g$ conv.

ACT: $f : [a,\infty) \to \mathbb{R}$. If $\int_a^\infty |f|$ conv. then $\int_a^\infty f$ conv.

(Converse doesn't hold: $\int_1^\infty \frac{\sin x}{x}\,dx$ conv, but $\int_1^\infty \frac{|\sin x|}{x}\,dx$ div.)

**Problem 3**
Determine whether the following integrals converge. Do not find their value.

1. $\displaystyle\int_0^\infty \frac{\sin(x)}{x^2+1}\,dx.$
2. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx.$
3. $\displaystyle\int_0^\infty \frac{1}{x+420}\,dx.$
4. $\displaystyle\int_0^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx.$
5. $\displaystyle\int_0^\infty \frac{3x}{x^3+x+2}\,dx.$
6. $\displaystyle\int_0^\pi \frac{\sin^2(x)}{\sqrt{x}}\,dx.$
7. $\displaystyle\int_0^1 \frac{1}{x^2+x}\,dx.$
8. $\displaystyle\int_0^1 \frac{1}{x^2+\sqrt{x}+2}\,dx.$

3. $$\int_0^\infty \frac{1}{x+420}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x+420}\,dx$$
$$= \lim_{b\to\infty}\int_{420}^{b+420} \frac{1}{u}\,du$$
$$= \int_{420}^\infty \frac{1}{u}\,du = \infty$$

2. $$\int_0^\infty \frac{x}{x^3+1}\,dx = \underbrace{\int_0^1 \frac{x}{x^3+1}\,dx}_{\text{proper integral} < \infty} + \underbrace{\int_1^\infty \frac{x}{x^3+1}\,dx}$$
$$\int_1^\infty \frac{x}{x^3+1}\,dx \le \int_1^\infty \frac{x}{x^3}\,dx = \int_1^\infty \frac{1}{x^2}\,dx < \infty$$
Conv. by p-test.

Thus $\int_0^\infty \frac{x}{x^3+1}\,dx$ conv.

5. $$\int_0^\infty \frac{3x}{x^3+x+2}\,dx = \underbrace{\int_0^1 \frac{3x}{x^3+x+2}\,dx}_{\text{proper}} + \underbrace{\int_1^\infty \frac{3x}{x^3+x+2}\,dx}$$
$$\int_1^\infty \frac{3x}{x^3+x+2}\,dx \le \int_1^\infty \frac{3x}{x^3} = 3\int_1^\infty \frac{1}{x^2}$$
Conv. by p-test.

1. $\int_0^\infty \frac{\sin(x)}{x^2+1}\,dx$ by ACT, if $\int_0^\infty \frac{|\sin(x)|}{x^2+1}\,dx$ conv., then $\int_0^\infty \frac{\sin(x)}{x^2+1}\,dx$ conv.

$$\int_0^\infty \frac{|\sin(x)|}{x^2+1}\,dx = \underbrace{\int_0^1 \frac{|\sin(x)|}{x^2+1}\,dx}_{\text{proper}} + \underbrace{\int_1^\infty \frac{|\sin(x)|}{x^2+1}\,dx}_{< \int_1^\infty \frac{1}{x^2}\,dx}$$
Conv. by p-test.

4. $$\int_0^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx = \int_0^1 \frac{x+1}{\sqrt{x^4-x}}\,dx + \underbrace{\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx}$$
$$\ge \int_1^\infty \frac{x}{\sqrt{x^4}}\,dx = \int_1^\infty \frac{1}{x}\,dx \quad \text{div.}$$

6. $\int_0^{\pi} \frac{\sin^2(x)}{\sqrt{x}}\,dx \leq \underbrace{\int_0^1 \frac{1}{\sqrt{x}}\,dx}_{\text{conv. by p-test}} + \underbrace{\int_1^{\pi} \frac{1}{\sqrt{x}}\,dx}_{\text{proper}}$

7. $\int_0^1 \frac{1}{x^2+x}\,dx$

$= \int_0^1 \frac{1}{x} - \frac{1}{x+1}\,dx$ $\qquad \frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$

$= \underbrace{\int_0^1 \frac{1}{x}\,dx}_{\text{div by p-test}} - \underbrace{\int_0^1 \frac{1}{x+1}\,dx}_{\text{proper}}$ So $\int_0^1 \frac{1}{x^2+x}\,dx$ div.

8. $\underbrace{\int_0^1 \frac{1}{x^2+\sqrt{x}+2}\,dx}_{\text{proper}}$

Define $\Gamma : (0,\infty) \to \mathbb{R}$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

1. Show that $\Gamma(1) = 1$.
2. Show that $\Gamma(x+1) = x\Gamma(x)$.
3. Conclude that $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \ldots$.

1. $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{b\to\infty} \int_0^b e^{-t}\,dt$

$= \lim_{b\to\infty} \left[-e^{-t} \Big|_{t=0}^{t=b}\right] = \lim_{b\to\infty} \left[-e^{-b}+1\right] = 1.$

2. $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt \quad \left( = \lim_{b\to\infty} \int_0^b t^x e^{-t}\,dt \right)$

$u = t^x \quad du = xt^{x-1}dt$
$dv = e^{-t}dt \quad v = -e^{-t}$

IBP $= \lim_{b\to\infty} \left( -t^x e^{-t} \Big|_0^b + \int_0^b x t^{x-1} e^{-t}\,dt \right)$

$= \underbrace{\lim_{b\to\infty} \left( -b^x e^{-b} \right)}_{=0} + x \int_0^\infty t^{x-1}e^{-t}\,dt = x\,\Gamma(x)$

$\lim_{b\to\infty} -b^x e^{-b} = -\lim_{b\to\infty} \frac{b^x}{e^b} \overset{L'H}{=} -\lim_{b\to\infty} \frac{xb^{x-1}}{e^b} \overset{LH}{=} \cdots$

$= -\lim_{b\to\infty} \frac{x(x-1)\cdots(x-k)\,b^{x-k-1}}{e^b}$ $\qquad -1 < x-k-1 \leq 0$

$= -\frac{\text{Something finite}}{\infty} = 0.$

3. Proof by induction.

$n=1$ $\quad \Gamma(n) = 1 = 0!$ ✓

Assume $\Gamma(n) = (n-1)!$

$\Gamma(n+1) = n\,\Gamma(n) = n\,(n-1)! = n!$ ✓