Tutorial 18

Problem 1

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijection such that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is also continuous.

1. Show that a sequence (a_n) converges if and only if the sequence $(f(a_n))$ converges. If $(f(a_n))$ converges to L, what must (a_n) converge to?
2. Give an example of a continuous function g and a sequence (b_n) such that $(g(b_n))$ converges but (b_n) does not converge.

Note: Assignment 12, q2c: the function *does not* have to be continuous everywhere; it only has to be continuous wherever the sequence (an) occurs.

1. Let's show $(f(a_n)) \xrightarrow{n\to\infty} f(L)$.

Let $\varepsilon>0$. WTS $\exists N \in \mathbb{N}$ $n>N \Rightarrow |f(a_n)-f(L)|<\varepsilon$.

Since f cts at L, $\exists \delta>0$ s.t. $|z-L|<\delta \Rightarrow |f(z)-f(L)|<\varepsilon$.

Since $(a_n) \xrightarrow{n\to\infty} L$, $\exists N \in \mathbb{N}$ $n>N \Rightarrow |a_n-L|<\delta$

$\Rightarrow |f(a_n)-f(L)|<\varepsilon$

as needed.

2. $a_n = (-1)^n$ (a_n) does not converge

Let $g(a)=0$ (continuous)

$g(a_n)=0 \xrightarrow{n\to\infty} 0$

Problem 2

1. Suppose $(a_n), (b_n), (c_n)$ are sequences of real numbers, such that for all $n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$. If (a_n) and (c_n) converge and
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$
show that (b_n) converges and $\lim_{n\to\infty} b_n = L$.
2. Show the same is true if we replace "for all $n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$" with "there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $a_n \leq b_n \leq c_n$."

1. Let $\varepsilon>0$. Want a $N\in\mathbb{N}$ s.t. $n>N \Rightarrow |b_n-L|<\varepsilon$.

Since $(a_n)\xrightarrow{n\to\infty} L$, $\exists N_a \in \mathbb{N}$ s.t. $n>N \Rightarrow |a_n-L|<\varepsilon$

$\Rightarrow L-\varepsilon<a_n$

Since $(c_n)\xrightarrow{n\to\infty} L$, $\exists N_c \in \mathbb{N}$ s.t. $n>N \Rightarrow |c_n-L|<\varepsilon$

$\Rightarrow c_n<L+\varepsilon$

Let $N=\max\{N_a, N_c\}$. if $n>N$ then

$n>N_a \Rightarrow L-\varepsilon<a_n\leq b_n$

$n>N_c \Rightarrow b_n\leq c_n<L+\varepsilon$ so $L-\varepsilon\leq b_n\leq L+\varepsilon \Rightarrow |b_n-L|<\varepsilon$.

2. Instead of $N=\max\{N_a,N_c\}$, Let $N=\max\{M,N_a,N_c\}$

Motivation: $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ (sometimes it's defined this way)

Problem 3

1. Prove that for all $x \in (0, \infty)$,

$$\lim_{n\to\infty} \frac{nx}{n+x} = x.$$

(*Hint*: Factor nx from the denominator.)

1. $\lim_{n\to\infty} \frac{nx}{n+x} = \lim_{n\to\infty} \frac{1}{\frac{1}{x}+\frac{1}{n}}$

$= \frac{1}{\frac{1}{x}+\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{\frac{1}{x}} = x$

Alternative: use L'Hôpital's.

2. Prove that, for all $x \in (0, \infty)$,

$$\log\left[\left(1+\frac{x}{n}\right)^n\right] = n\int_n^{n+x} \frac{1}{t}\, dt.$$

2. $\log\left[\left(1+\frac{x}{n}\right)^n\right] = n \log\left(1+\frac{x}{n}\right)$

$= n\int_n^{n+x} \frac{1}{t}\, dt.$

$\int_n^{n+x} \frac{1}{t}\, dt$

$\overset{FTC}{=} \log(n+x) - \log(n)$

$= \log\left(\frac{n+x}{n}\right) - \log\left(1+\frac{x}{n}\right)$

3. Prove that, for all $x \in (0, \infty)$,

$$\lim_{n\to\infty} \log\left[\left(1+\frac{x}{n}\right)^n\right] = x.$$

(*Hint*: use problem 1 and 2.)

Problem 1: $\lim_{n\to\infty} \frac{nx}{n+x} = x$

Problem 2: $\log\left[\left(1+\frac{x}{n}\right)^n\right] = n\int_n^{n+x} \frac{1}{t}\, dt$

$\log\left[\left(1+\frac{x}{n}\right)^n\right] = n\int_n^{n+x} \frac{1}{t}\, dt$

$\le n\int_n^{n+x} \frac{1}{n}\, dt = n\cdot\frac{x}{n} = x \xrightarrow{n\to\infty} x.$

$\log\left[\left(1+\frac{x}{n}\right)^n\right] = n\int_n^{n+x} \frac{1}{t}\, dt$

$\ge n\int_n^{n+x} \frac{1}{n+x}\, dt = \frac{nx}{n+x} \xrightarrow{n\to\infty} x.$

Thus, by squeeze, $\lim_{n\to\infty} \log\left[\left(1+\frac{x}{n}\right)^n\right] = x.$

4. Compute $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. (*Hint*: use problem 1.)

in tutorial, not in problem 3.

$\lim_{n\to\infty} \log\left[\left(1+\frac{x}{n}\right)^n\right] = x.$

Since $x \mapsto e^x$ cts,

$\lim_{n\to\infty} \exp\left[\log\left[\left(1+\frac{x}{n}\right)^n\right]\right] = \exp(x)$ (by Problem 1)

$\parallel$

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

Problem 4

Suppose (a_n) is a bounded sequence of real numbers. Define

$$(s_n) = \sup_{k\geq n} a_k = \sup\{a_k : k \geq n\},$$
$$(t_n) = \inf_{k\geq n} a_k = \inf\{a_k : k \geq n\}.$$

1. Show that (s_n) is monotone nonincreasing and (t_n) is monotone nondecreasing. Conclude that they both converge. regardless of whether (a_n) converges or not!

   Remark: We write

$$\lim_{n\to\infty} s_n = \limsup_{n\to\infty} a_n,$$
$$\lim_{n\to\infty} t_n = \liminf_{n\to\infty} a_n.$$

   They are called the "limit superior" and "limit inferior" of (a_n), respectively.

if $m > n$, then $\{a_k : k \geq m\} \subseteq \{a_k : k \geq n\}$

$\sup\{a_k : k \geq m\} \leq \sup\{a_k : k \geq n\}$

$s_m \leq s_n$

So (s_n) nonincreasing. Similarly (t_n) nondecreasing.

(s_n) nonincreasing, bounded from below $\overset{MCT}{\Rightarrow}$ (s_n) converges.

(a_n) is bounded

Same with (t_n).

2. Show that (a_n) converges to $L \in \mathbb{R}$ if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L.$$

$(\Rightarrow)$ Suppose $(a_n) \to L$. Let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n - L| < \frac{\varepsilon}{2} \Rightarrow a_n < L + \frac{\varepsilon}{2}$ and $a_n > L - \frac{\varepsilon}{2}$

if $n > N$, then $s_n = \sup_{k\geq n} a_k \leq L + \frac{\varepsilon}{2} < L + \varepsilon$.

$a_k < L + \frac{\varepsilon}{2}$ if $k \geq n$

(lim inf proof similar).

$s_n = \sup_{k\geq n} a_k > L - \frac{\varepsilon}{2}$. $\longrightarrow$ $L - \frac{\varepsilon}{2} < s_n \leq L + \frac{\varepsilon}{2}$.

$a_k > L - \frac{\varepsilon}{2}$ for all $k \geq n$

$|s_n - L| < \varepsilon$.

($\Leftarrow$) Suppose $(s_n) \to L$, $(t_n) \to L$.

$\inf_{k \geq n} a_n = t_n \leq a_n \leq s_n = \sup_{k \geq n} a_n$

by Squeeze, $(a_n) \to L$.

Problem 5
Fix some number $x > 0$. Define a sequence (a_n) as follows: let $a_1 = 1$, and for all $n > 1$ define a_n recursively as

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{x}{a_{n-1}}\right).$$

1. Prove that for all $n \in \mathbb{N}$, $a_n > 0$.

1. Proof by induction.

$n=1$: $a_1 = 1 > 0$

Assume $a_n > 0$. $a_{n+1} = \frac{1}{2}(\underbrace{a_n}_{>0} + \overbrace{\frac{x}{\underbrace{a_n}_{>0}}}^{>0}) > 0$.

2. Prove that for all $t \in (0, \infty)$,

$$t + \frac{1}{t} \geq 2.$$

Use this to show that for $n \geq 2$, $a_n \geq \sqrt{x}$. (*Hint*: factor out $\sqrt{x}$ from the parentheses.)

$t + \frac{1}{t} \geq 2 \Leftrightarrow t^2 + 1 \geq 2t \Leftrightarrow t^2 - 2t + 1 \geq 0 \Leftrightarrow (t-1)^2 \geq 0$

For $n \geq 2$, $a_n = \frac{1}{2}(a_{n-1} + \frac{x}{a_{n-1}})$

$= \frac{1}{2}\sqrt{x}\left(\frac{a_{n-1}}{\sqrt{x}} + \frac{\sqrt{x}}{a_{n-1}}\right)$

$\geq \frac{1}{2}\sqrt{x}(2)$ (with $t = \frac{a_{n-1}}{\sqrt{x}}$)

$= \sqrt{x}$.

3. Prove that (a_n) is non-increasing for $n \geq 2$. Conclude that (a_n) converges. (*Hint*: prove that for $n \geq 2$, $\frac{a_{n+1}}{a_n} \leq 1$.)

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2}(a_n + \frac{x}{a_n})}{a_n} = \frac{1}{2}\left(1 + \frac{x}{a_n^2}\right) \leq \frac{1}{2}\left(1 + \frac{x}{\sqrt{x}^2}\right) = 1$. (a_n) converges since it's non increasing and bounded by $\sqrt{x}$ from below. MCT

4. Compute $\lim_{n\to\infty} a_n$.

Since (a_n) converges, we know $\lim_{n\to\infty} a_n$ exists.

Let $c = \lim_{n\to\infty} a_n = \frac{1}{2}\lim_{n\to\infty}(a_{n-1} + \frac{x}{a_{n-1}})$

$= \frac{1}{2}\left[(\lim_{n\to\infty} a_{n-1}) + \frac{x}{\lim_{n\to\infty} a_{n-1}}\right]$ $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_n$

$= \frac{1}{2}(c + \frac{x}{c})$

$c = \frac{1}{2}c + \frac{1}{2}\frac{x}{c} \Rightarrow c^2 = \frac{1}{2}c^2 + \frac{1}{2}x$

$\Rightarrow \frac{1}{2}c^2 = \frac{1}{2}x \Rightarrow c = \pm\sqrt{x}$.

but $c = \sqrt{x}$ since (a_n) positive