Tutorial 19

Problem 1

1. Suppose that $\sum_{n=0}^{\infty} a_n^2$ converges. Must it also be the case that $\sum_{n=0}^{\infty} a_n$ converges?

2. Suppose that $\sum_{n=0}^{\infty} a_n$ converges. Must it also be the case that $\sum_{n=0}^{\infty} a_n^2$ converges?

1. No. $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ converges (p-test) but $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

2. No. $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}$ converges by Alternating Series Test,
but $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

Note: if f cts at L and $(a_n) \longrightarrow L$,
then $f(a_n) \to f(L)$.

Here, $(S_n) \longrightarrow L$
but $f(S_n) \neq \sum_{k=0}^{n} f(a_n)$

Convergent $\Rightarrow$ Cauchy $\leftarrow$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ $m, n > N \Rightarrow |a_m - a_n| < \varepsilon$

Not Cauchy $\Rightarrow$ divergent.

Problem 2

Suppose that (a_n) is a sequence of *positive* real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Let $r_n = \sum_{m=n}^{\infty} a_m$. Prove that if $m < n$,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

1. Let $S_n = \sum_{k=1}^{n} a_k$. S_n increasing (since a_n positive).
2. r_n decreasing. Let $S = \sum_{n=1}^{\infty} a_n$, then $r_n = \sum_{K=1}^{\infty} a_K - \sum_{K=1}^{n-1} a_K = S - S_{n-1}$
3. $\sum_{k=1}^{n} \frac{a_k}{r_k}$ not Cauchy, then $\sum_{k=1}^{\infty} \frac{a_k}{r_k}$ diverges.

$$\frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdots + \frac{a_n}{r_n} > \frac{a_m + a_{m+1} + \cdots + a_n}{r_m} > 1 - \frac{r_n}{r_m}$$

r_n decreasing

$\Leftrightarrow$ $a_m + a_{m+1} + \cdots + a_n > r_m - r_n = \sum_{k=m}^{\infty} a_k - \sum_{k=n}^{\infty} a_k$

$= a_m + a_{n+1} + \cdots + a_{n-1}$

Indeed, $a_m + \cdots + a_n > a_m + \cdots + a_{n-1}$

Since $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$

$\sum_{k=1}^{\infty} \frac{a_k}{r_k}$ Let $T_n = \sum_{k=1}^{n} \frac{a_k}{r_k}$.

WTS (T_n) not Cauchy

Observe: $\forall m \in \mathbb{N}$, $\lim_{n\to\infty} \frac{r_n}{r_m} = 0$.

(Since $r_n = \sum_{k=n}^{\infty} a_k \xrightarrow{n\to\infty} 0$ since $\sum_{k=1}^{\infty} a_k$ converges.)

So $\forall m \in \mathbb{N}$, $\lim_{n\to\infty} 1 - \frac{r_n}{r_m} = 1$

For $n > m$, $r_n < r_m$ so $1 - \frac{r_n}{r_m}$ increasing with n.

$T_{n+1} - T_m$

So: $\sum_{k=m}^{n} \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m} > \frac{1}{2}$ (Choosing any m, if n large enough)

Let $\varepsilon = \frac{1}{2}$. $\nexists N \in \mathbb{N}$ s.t. $|T_{n+1} - T_m| < \frac{1}{2}$. $\forall n, m > N$.

(T_n) not cauchy $\Rightarrow$ (T_n) doesn't converge

Solution. Define $s_1 = 0$ and $s_n = \sum_{i=1}^{n-1} a_n$ where $n > 1$, and let $S = \sum_{n=1}^{\infty} a_n$. Note that since (a_n) is positive, (s_n) is increasing, so $(r_n) = (S - s_n)$ is decreasing and positive. Thus

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} = \frac{a_m + \cdots + a_n}{r_m} > \frac{a_m + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}.$$

Note that $1 - \frac{r_n}{r_m} > 0$ because (r_n) is decreasing. Now, note that $\lim_{n\to\infty} 1 - \frac{r_n}{r_m} = 1$, so $\sum_{i=m}^{n} \frac{a_i}{r_i} \geq 1$. Thus, the sequence of partial sums is not Cauchy and the series does not converge. $\square$

Problem 3
Using the identity for geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

for $|x| < 1$, express the following series as expressions not involving summations, and state for which x they converge.

1. $\sum_{k=0}^{\infty}(-1)^k x^{2k} = \sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1+x^2}$ need $|-x^2|<1$, or $|x|<1$.
2. $\sum_{k=0}^{\infty}(1-x)^{3k} = \sum_{k=0}^{\infty}((1-x)^3)^k = \frac{1}{1-(1-x)^3}$ need $|(1-x)^3|<1$, or $|1-x|<1$ or $0<x<2$.
3. $\sum_{k=0}^{\infty}\frac{1}{(x-2)^k} = \sum_{k=0}^{\infty}\left(\frac{1}{x-2}\right)^k = \frac{1}{1-\frac{1}{x-2}}$ need $\left|\frac{1}{x-2}\right|<1$

$1 < |x-2|$

$x>3$ or $x<1$.

Problem 4
Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

Let $S_n = \sum_{k=1}^{n} a_k$.

$|S_n| = \left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$.

$\lim_{n\to\infty} |S_n| \leq \lim_{n\to\infty} \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{\infty} |a_n|$ exists cuz $\sum_{k=1}^{\infty} a_k$ abs. conv.

$\lim_{n\to\infty} |S_n| = \left|\lim_{n\to\infty} S_n\right| = \left|\lim_{n\to\infty} \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=1}^{\infty} a_k\right|$.