Tutorial 2: Suprema and Infima

• (Problem 3) if max S exists, then max S = sup S. but max S might not exist.
• sup S might not be in S, but max S has to be in S.
• sup S exists as long as $S \neq \emptyset$ and has an upper bound but max S might not exist, even if sup S exists.

Problem 0
Briefly discuss the difference between *supremum* and *maximum*.

Let $S \subseteq \mathbb{R}$. We call a real number x an *upper bound* for S if

$$(\forall s \in S)[x \geq s].$$

The *completeness axiom* states that if $S \subseteq \mathbb{R}$ is nonempty and has an upper bound, then it has a least upper bound, called the *supremum* of S. That is, there is a real number M such that:

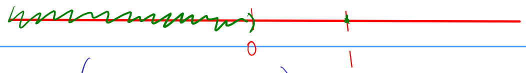
1. $M \geq s$ for all $s \in S$; ← S is an upper bound
2. For any upper bound x of S, we have $x \geq M$. ← least upper bound

In fact, M is unique (see Problem 2). We use $\sup S$ to denote the[1] supremum of S.

Problem 1
For each of the following sets, find the supremum/infimum or show that it doesn't exist. Which sets have maxima/~~infima~~ minima?

1. $(-\infty, 0) \cup \{1\}$.
2. $\mathbb{Q} \cap (-\infty, \sqrt{2})$.
3. $\{\frac{1}{n} : n \in \mathbb{N}\}$.

(1) $(-\infty, 0) \cup \{1\}$



$\sup\left((-\infty,0) \cup \{1\}\right) = 1$.

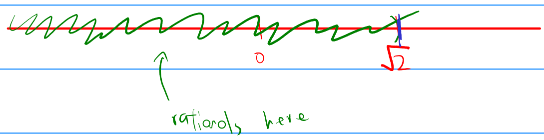
inf doesn't exist.

$\max\left((-\infty,0) \cup \{1\}\right) = 1$.

↑ has to be in the set itself

no minimum.

(2) $\mathbb{Q} \cap (-\infty, \sqrt{2})$



$\sup\left(\mathbb{Q} \cap (-\infty, \sqrt{2})\right) = \sqrt{2}$.

$\sqrt{2}$ is an upper bound: $x \leq \sqrt{2}$ for any $x \in \mathbb{Q} \cap (-\infty, \sqrt{2})$

$\sqrt{2}$ is the least upper bound:

if $N < \sqrt{2}$, there exists a rational $\frac{p}{q}$ s.t. $N < \frac{p}{q} < \sqrt{2}$
(rationals are dense).

so N is not an upper bound.

max doesn't exist!

min doesn't exist.

Problem 2
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Show that the supremum of S is unique: if both M and N are least upper bounds for S, then $M = N$.

Suppose M and N are both least upper bounds of S.

M least upper bound of S: ① $M \geq s \quad \forall s \in S$
② $M \leq x$ for any upper bound x of S.

N least upper bound of S: ③ $N \geq s \quad \forall s \in S$
④ $N \leq x$ for any upper bound x of S.

Since N is an upper bound for S, by ②
$M \leq N$.

Since M is an upper bound for S, by ④
$N \leq M$.

So $M = N$.

Problem 3
Let $S \subseteq \mathbb{R}$ have a maximum M. Show that $M = \sup S$.

$(\forall s \in S)\ M \geq s$ (and $M \in S$)

① M is an upper bound
$(\forall s \in S)\ M \geq s$ (since $M = \max S$)

10:49

② M is the least upper bound.
Let N be an upper bound for S. WTS $M \leq N$.
if (towards contradiction) $N < M$
then $N <$ something in S. (since $M \in S$).
So N is not an upper bound
Thus, $M \leq N$.

Problem 4
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Show that $M = \sup S$ if and only if M is an upper bound for S and $(\forall \epsilon > 0)(\exists s \in S)[M < s + \epsilon]$. *Note: This is hard. Try taking the contrapositive of both implications.*

P Q

$(\Rightarrow)$ ~~Let $M = \sup S$. WTS $(\forall \varepsilon > 0)(\exists s \in S)[M < s + \varepsilon]$~~ and M is an upper bound

$\neg Q$ $\neg P$

M is not an upper bound or $(\exists \varepsilon > 0)(\forall s \in S)[M \geq s + \varepsilon]$. WTS $M \neq \sup S$.

Then $M - \varepsilon$ is an upper bound for S. $(\forall s \in S)[M - \varepsilon \geq s]$

not least upper bound.

and $M - \varepsilon < M$ so M can't be the least upper bound.

$(\Leftarrow)$ ~~Assume M is an upper bound for S, and $(\forall \varepsilon>0)(\exists s\in S)[M<s+\varepsilon]$. WTS $M=\sup S$.~~

Assume $M \neq \sup S$. WTS M is <u>not</u> an upper bound for S, or $(\exists \varepsilon>0)(\forall s\in S)[M \geq s+\varepsilon]$.

$M \neq \sup S$: either: M is not an upper bound, or, there is some upper bound N of S s.t. $N<M$

if $N<M$, N is an upper bound of S

Let $\varepsilon = M-N$.

$(\forall s\in S)[N \geq s]$

$(\forall s\in S)[N+\varepsilon \geq s+\varepsilon]$

$(\forall s\in S)[M \geq s+\varepsilon]$

So $(\exists \varepsilon>0)(\forall s\in S)[M \geq s+\varepsilon]$

Question: What are the upper bounds of ϕ? Does $\sup \phi$ exist?

Ans: <u>Any</u> number $M\in\mathbb{R}$ is an upper bound for ϕ.

$(\forall s\in\phi)[M\geq s]$ ✓

There is no l.u.b.

Problem 5
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above, and $c>0$. Define

$$cS = \{cs : s\in S\}.$$

Show that $\sup cS = c\sup S$.

<u>Ex</u> if $S=[-2,5)$ then $cS=[-2c,5c)$.

Part 1: show $\sup cS$ exists.

cS is nonempty: if $s\in S$ then $cs\in cS$.

there exists such an S as $S\neq\phi$.

cS is bounded from above: S is bounded from above by x.

$(\forall s\in S)[s\leq x]$.

$(\forall t\in cS)[t\leq cx]$

So cS is bounded from above.

Part 2: $\sup cS = c \sup S$. Let $M = \sup cS$, $N = \sup S$.

We show ① $M \le cN$

and ② $M \ge cN$.

① cN is an upper bound for cS:

if $N \ge s$

then $cN \ge cs$

Since M is the least upper bound of cS,

$M \le cN$.

② We show $\frac{1}{c} M \ge N$.

$\frac{1}{c} M$ is an upper bound for S:

if $M \ge cs$ then $\frac{1}{c} M \ge s$.

Since N is the least u.b., $\frac{1}{c} M \ge N$.

Problem 6
Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Define

$$-S = \{-s : s \in S\}.$$

Show that $\inf -S = -\sup S$.

Part 1. $\inf -S$ exists:

$-S$ is nonempty: if $s \in S$ then $-s \in -S$.

such an s exists.

$-S$ is bounded from below.

if $(\forall s \in S)[x \ge s]$ then $(\forall -s \in S)(-x \le -s)$

$-x$ bounds $-S$ from below.

Part 2: Let $M = \inf -S$, $N = \sup S$.

① $M \le -N$

② $M \ge -N$

① We prove $-M \ge N$

$-M$ is an upper bound for S:

since M is a lower bound for $-S$,

$(\forall -s \in -S)(M \le -s)$

$(\forall s \in S)(-M \ge s)$.

Since N is the least u.b,

$-M \ge N$.

② $-N$ is a lower bound for $-S$:

$(\forall s \in S)\ [N \geq s]$

$(\forall -s \in -S)\ [-N \leq -s]$

Since M is the greatest lower bound,

$M \geq -N$.

Problem 7
Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. We can define a new function $f + g : \mathbb{R} \to \mathbb{R}$ given by $(f + g)(x) = f(x) + g(x)$. We can also define $\sup h$, for any function $h : \mathbb{R} \to \mathbb{R}$, as:

$$\sup h = \sup\{h(x) : x \in \mathbb{R}\}$$

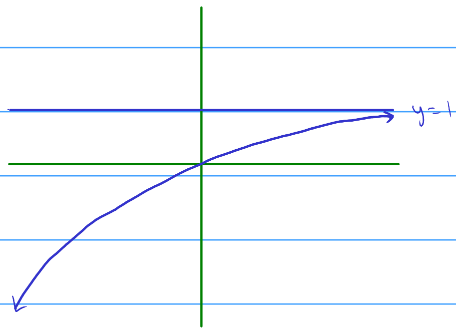
Prove that if $\sup f$ and $\sup g$ both exist, then

$$\sup(f + g) \leq \sup f + \sup g.$$

Give examples of f and g for which the above inequality is strict.

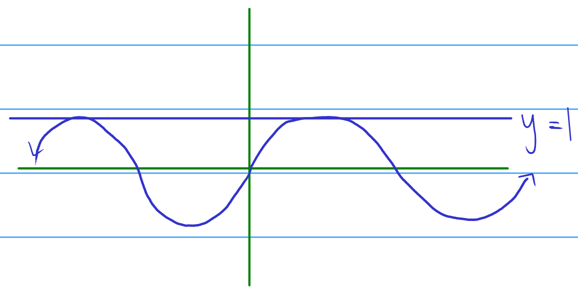
$f(x) = \sin x$
$g(x) = \cos x$
$(f+g)(x) = \sin x + \cos x$

Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - e^{-x}$



$\sup f = 1$

$g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin x$.



$\sup g = 1$