Problem 1

1. Suppose that f is continuous and that the sequence

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

converges to L. Prove that L is a "fixed point" for f, i.e., $f(L) = L$.

2. A function $f : [a, b] \to [a, b]$ is called a contraction if there exists $c < 1$ such that, for all $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq c|x - y|.$$

Prove that any contraction has a unique fixed point.

Solution. 1. Let a_n be f composed with itself n times, evaluated on x. Then $\lim_{n\to\infty} a_n = L$. Since f is continuous,

$$f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n),$$

but $f(a_n) = a_{n+1}$, so $\lim_{n\to\infty} f(a_n) = L$.

2. First, note that a contraction is continuous. To see this, let $\epsilon > 0$ be given. Then if $|x - y| < \epsilon/c$, $|f(x) - f(y)| \leq c|x - y| < \epsilon$. Now, we'll show that $x, f(x), f(f(x)), \ldots$ converges. Suppose $|f(x) - f(y)| < c|x - y|$ for all $x, y \in [a, b]$. Let $x \in [a, b]$ and $\epsilon > 0$ be given. Since $0 < c < 1$, we know $c^n \to 0$ as $n \to \infty$. Thus, there exists $N \in \mathbb{N}$ such that $n > N$ implies $c^n < \dfrac{\epsilon(1-c)}{|x - f(x)|}$. Now let $m > n > N$. We have

$$\begin{aligned}
|f^m(x) - f^n(x)| &= |f^m(x) + (f^{m-1}(x) - f^{m-1}(x)) + \cdots + (f^{n+1}(x) - f^{n+1}(x)) - f^n(x)| \\
&\leq |f^m(x) - f^{m-1}(x)| + \cdots + |f^{n+1}(x) - f^n(x)| \\
&\leq c^{m-1}|x - f(x)| + \cdots + c^{n+1}|x - f(x)| + c^n|x - f(x)| \\
&= c^n|x - f(x)| \cdot \sum_{k=0}^{m-n-1} c^k \\
&\leq c^n|x - f(x)| \cdot \sum_{k=0}^{\infty} c^k \\
&= c^n|x - f(x)| \cdot \left(\frac{1}{1-c}\right) \\
&< \frac{\epsilon(1-c)}{|x - f(x)|} \cdot |x - f(x)| \cdot \left(\frac{1}{1-c}\right) \\
&= \epsilon.
\end{aligned}$$

Thus, this sequence is Cauchy and thus convergent. By part 1, the limit L of this sequence is a fixed point of f. If L' were a different fixed point, we could let $\epsilon > 0$ be given, and let $N \in \mathbb{N}$ be so large that $n > N$ implies $c^n < \epsilon/|L - L'|$. Then $|L - L'| = |f^n(L) - f^n(L)| \leq c^n|L - L'| < \epsilon$. Since $\epsilon > 0$ is arbitrary, $L = L'$; a contradiction. Thus the fixed point of f is unique.

□

Recall that if $(s_n), (t_n)$ are sequences of real numbers, and for some $N \in \mathbb{N}$ we have $n > N$ implies $s_n \leq t_n$, then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

and for any bounded sequence (a_n), $\lim_{n\to\infty} a_n = L$ if and only if

$$\liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} a_n.$$

Also recall the **binomial theorem**: for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Problem 2

1. Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.
2. Prove that for all $n \in \mathbb{N}$,
$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$
Conclude that $\displaystyle\left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!}$.
3. Prove that
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq \sum_{n=0}^{\infty} \frac{1}{n!}.$$
Conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$. *Hint*: Fix some $m \in \mathbb{N}$ and show $\displaystyle\sum_{k=0}^{m} \frac{1}{k!} \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, then let m go to infinity.
4. Show that for all $n \in \mathbb{N}$,
$$e - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{m=n+1}^{\infty} \frac{1}{m!} < \frac{1}{n!n}.$$
How many terms of this series do we need to compute e accurate up to 10 decimal places?
5. Prove that e is irrational. *Hint*: If $e = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then
$$0 < q!q\left(e - \sum_{k=0}^{q} \frac{1}{k!}\right) < 1.$$
How does this lead to a contradiction?

Solution. 1. Notice that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} &= 1 + 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \cdots \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\end{aligned}$$

so this series converges by the basic comparison test. (Alternatively, the RHS on the last line sums to 3 and the sequence of partial sums is increasing.)

2. By the Binomial Theorem,

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \frac{(n-1)\cdots(n-(k-1))}{n^{k-1}} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-(k-1)}{n}\right) \\ &= \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right).\end{aligned}$$

Note that since $k \leq n$, $0, \ldots, k-1 < n$, so $\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) < 1$ meaning

$$\sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) < \sum_{k=0}^{n} \frac{1}{k!}$$

3. Suppose $m < n$ are natural numbers, then

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \\ &> \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right),\end{aligned}$$

Taking the limit of both sides as $n \to \infty$,

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^{m} \frac{1}{k!}.$$

Taking the limit of both sides as $m \to \infty$,

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^{\infty} \frac{1}{k!}$$

4.

$$
\begin{aligned}
0 < e - \sum_{k=0}^{n} \frac{1}{k!} &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots \\
&= \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \cdots\right] \\
&< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} \cdots\right] \\
&= \frac{1}{(n+1)!}\left[\frac{1}{1 - \frac{1}{n+1}}\right] \\
&= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} \\
&= \frac{1}{n!n}
\end{aligned}
$$

Note that $13! \cdot 13 = 80951270400 > 8 \cdot 10^{10} > 10^{10}$, so $\dfrac{1}{13! \cdot 13} < 10^{-10}$.

5. If $e = p/q$.

$$0 < q!q\left(e - \sum_{k=0}^{q} \frac{1}{k!}\right) < 1.$$

Since $qe = p$, $q!qe \in \mathbb{Z}$. Since $q \geq 0, \ldots, q$, for all $k = 0, \ldots, q$, $q!/k! \in \mathbb{Z}$, so $q!q\displaystyle\sum_{k=0}^{q} \frac{1}{k!} \in \mathbb{Z}$. But there are no integers strictly between 0 and 1 so we have a contradiction. Thus, e is irrational.

$\square$

Problem 3
Show that if $\lim_{n\to\infty} a_n = L$ then

$$\lim_{n\to\infty} \frac{(a_1 + \cdots + a_n)}{n} = L.$$

Hint: Separate and bound: for large enough N, a_N is close to L. This means $a_N + \cdots + a_{N+M}$ is close to $M \cdot L$, so $(a_N + \cdots + a_{N+M})/(N+M)$ is close to $(M \cdot L)/(M+N)$. If M is large in comparison to N, then $(M \cdot L)/(M+N)$ is close to L.

Solution. Let $\epsilon > 0$ be given and let N be large enough so that $n > N$ implies $|a_n - L| < \epsilon$. Then for all $M \in \mathbb{N}$,

$$\begin{aligned}\left|\left(\frac{1}{N+M}\sum_{k=1}^{N+M} a_k\right) - L\right| &\leq \frac{1}{N+M}\left|\sum_{k=1}^{N} a_k - NL\right| + \left|\frac{1}{N+M}\sum_{k=N+1}^{N+M} a_k - \frac{ML}{N+M}\right| \\ &< \frac{1}{N+M}\left|\sum_{k=1}^{N} a_k - NL\right| + \frac{M\epsilon}{N+M},\end{aligned}$$

Note, though, that $1/(N+M) \to 0$ and $M/(N+M) \to 1$ as $M \to \infty$. So assume M' is large enough s.t. $M > M'$ implies $1/(N+M) < \dfrac{\epsilon}{\left|\sum_{k=1}^{N} a_k - NL\right|}$ (if the expression in absolute value is positive; otherwise there's nothing to do here) and $M(N+M) < 1 + \epsilon$, then if $n = N + M$,

$$\left|\frac{(a_1 + \cdots + a_n)}{n} - L\right| < \frac{1}{N+M}\left|\sum_{k=1}^{N} a_k - \frac{NL}{N+M}\right| + \frac{M\epsilon}{N+M} < 2\epsilon + \epsilon^2.$$ □