Tutorial 21

Let $(f_n : I \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of functions. If $f : I \to \mathbb{R}$ is another function, we say that f_n converges uniformly to f is for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for every $x \in \mathbb{R}$ we have $|f_n(x) - f(x)| < \epsilon$.

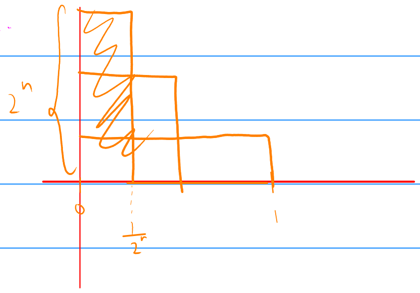
We say that f_n converges pointwise to f is for every $x \in \mathbb{R}$, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.

Problem 1

1. Provide a sequence of continuous functions $f_n : I \to \mathbb{R}$ that converges pointwise to f, yet f is not continuous.
2. Provide a sequence of integrable functions $f_n : I \to \mathbb{R}$ that converges pointwise to an integrable function f, yet the sequence $(\int f_n)$ does not converge to $\int f$.
3. Provide a sequence of integrable functions $f_n : I \to \mathbb{R}$ that converge pointwise to a non-integrable function f.
4. Provide a sequence of nonnegative bounded functions $f_n : I \to \mathbb{R}$, with $M_n = \sup f_n$, such that $\sum_{n=1}^{\infty} f_n$ converges uniformly, yet $\sum_{n=1}^{\infty} M_n$ does not converge. This shows the converge to the Weierstrass M-test does not hold.

1. $f_n(x) = x^n$ over $[0,1]$

$f_n \xrightarrow{\text{ptwise}} f \qquad f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$

2. 

$f_n(x) = \begin{cases} 2^n & x \in [0, \frac{1}{2^n}] \\ 0 & x \in [\frac{1}{2^n}, 1] \end{cases}$

$\int_0^1 f = 1$

but $f \xrightarrow{\text{ptwe}} 0 \qquad \int 0 = 0$.

3. $[0,1]$

$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad f_n \xrightarrow{\text{ptwe}} f$

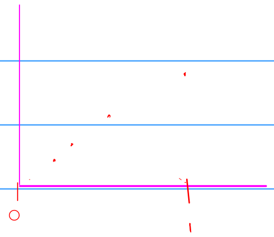
$f_n(x) = \begin{cases} 1 & x = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z},\ 0 < q \leq n \\ 0 & \text{o.w.} \end{cases}$

each f_n has finitely many discontinuities

hence integrable.

4. Provide a sequence of nonnegative bounded functions $f_n : I \to \mathbb{R}$, with $M_n = \sup f_n$, such that $\sum_{n=1}^{\infty} f_n$ converges uniformly, yet $\sum_{n=1}^{\infty} M_n$ does not converge. This shows the converge to the Weierstrass M-test does not hold.

4.



$$f_n(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n} \\ 0 & \text{o.w.} \end{cases}$$

$$\sum f_n(x) \xrightarrow{\text{unif}} f(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{o.w.} \end{cases}$$

$M_n = \frac{1}{n}$ $\quad \sum M_n$ diverges.

Problem 2
Suppose that f_n are continuous functions on $[0,1]$ that converge uniformly to f. Prove that

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n = \int_0^1 f.$$

Is this true if the convergence isn't uniform?

Let $\varepsilon > 0$. Want N s.t. $n > N \Rightarrow$

$$\left| \int_0^{1-1/n} f_n - \int_0^1 f \right| < \varepsilon.$$

$$\left| \int_0^{1-\frac{1}{n}} f_n - \int_0^1 f \right| = \left| \int_0^{1-\frac{1}{n}} f_n - \int_0^{1-\frac{1}{n}} f - \int_{1-\frac{1}{n}}^1 f \right|$$

$$\leq \left| \int_0^{1-\frac{1}{n}} f_n - f \right| + \left| \int_{1-\frac{1}{n}}^1 f \right|$$

$$\leq \int_0^{1-\frac{1}{n}} |f_n - f| + \left| \int_{1-\frac{1}{n}}^1 f \right|$$

$$\leq \int_0^1 |f_n - f| + \left| \int_{1-\frac{1}{n}}^1 f \right|$$

Find N_1 s.t. $n > N_1$, then $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ $\forall x \in [0,1]$

(using uniform conv.)

Since $f_n \xrightarrow{\text{unif}} f$ and f_n cts, f cts.

Thus $\lim_{n\to\infty} \int_{1-\frac{1}{n}}^1 f = 0$

Find N_2 s.t. $n > N_2$ $\left| \int_{1-\frac{1}{n}}^1 f \right| < \frac{\varepsilon}{2}$.

Then $N = \max\{N_1, N_2\}$

$$n > N \Rightarrow \left| \int_0^{1-\frac{1}{n}} f_n - \int_0^1 f \right| \leq \int_0^1 |f_n - f| + \left| \int_{1-\frac{1}{n}}^1 f \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Problem 3
Prove that for $-1 < x \le 1$,

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Hint : 1. if $f_n \xrightarrow{\text{unif}} f$, f_n, f integrable then $\int f_n \to \int f$

2. $\log(1+x) = \int_1^x \frac{1}{t}\, dt$

3. Geometric series converges uniformly.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad -1<x<1$$

4. Integrate power series term-by-term.

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$\frac{1}{1+x} \overset{\text{power series diff.}}{=} \sum_{n=1}^{\infty} (-x)^{n-1} = \sum_{n=0}^{\infty} (-x)^n \qquad (-1<x<1)$$

geometric series

$$\int_0^x \frac{1}{1+t}\, dt \qquad \int_0^x \sum_{n=0}^{\infty} (-t)^n \, dt$$

$$= \log(1+x) - \log(1+0) \qquad \overset{\text{power series int.}}{=} \sum_{n=0}^{\infty} \int_0^x (-t)^n \, dt$$

$$= \log(1+x)$$

$$= \sum_{n=0}^{\infty} -\frac{1}{n+1} (-t)^{n+1} \Big|_{t=0}^{t=x}$$

$$= \sum_{n=0}^{\infty} -\frac{1}{n+1} (-x)^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+2} x^{n+1}}{n+1}$$

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad -1<x<1$$

for $x=1$

Problem 3
Prove that for $-1 < x \le 1$,

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Problem 4

1. Write down the power series for $\log(1-x)$ and $\log[(1+x)/(1-x)]$ around $x=0$.
2. Show that the power series for $f(x) = \log(1-x)$ converges only for $-1 \le x < 1$, and that the power series for $g(x) = \log[(1+x)/(1-x)]$ converges only for x in $(-1, 1)$.

1. $\log(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-x)^n}{n}$

$= -\sum_{n=1}^{\infty} \frac{x^n}{n}$

$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + \sum_{n=1}^{\infty} \frac{x^n}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n + x^n}{n}$

2. Radius of conv. for $\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$

if $|x|<1$, $\left|\frac{x^n}{n}\right| < \left|\frac{1}{n}\right|^n$

Ratio test: $\lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} x\right|$

$= |x| < 1.$

by Ratio test, $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $|x|<1$.

Radius of conv ≥ 1.

Radius of conv ≤ 1 (apply ratio test to $|x|>1$).

$x \in (-1, 1)$ converges

$x \in \mathbb{R} \setminus [-1, 1]$ diverges.

$x=1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x=-1$: $-\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST.

So $-\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges for $x \in [-1, 1)$.

$\log\left(\frac{1+x}{1-x}\right)$: Similar