Problem 1

For $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, we define[a]

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}.$$

In this problem, we will deduce that for $|x| < 1$,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \tag{1}$$

1. Show that $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ converges if $|x| < 1$ using the ratio test.
2. Let $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ for $|x| < 1$. Show that $(1+x)f'(x) = \alpha f(x)$.
3. Suppose $f : (-1, 1) \to \mathbb{R}$ is a differentiable function satisfying $(1+x)f'(x) = \alpha f(x)$. Show that $f(x) = c(1+x)^\alpha$ for some constant c.

   Hint: Consider $g(x) = f(x)/(1+x)^\alpha$.
4. Conclude (1), i.e. $c = 1$ in the previous subproblem.

[a] This extends the definition of the binomial $\binom{m}{n}$ for $m \in \mathbb{N}, 0 \leq n \leq m$.

Solution

1. Let $a_n = \binom{\alpha}{n} x^n$. We have

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{\frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!} x^{n+1}}{\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\alpha-n}{n+1} x \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{\alpha}{n+1} - \frac{n}{n+1} \right| \\
&= |x| \cdot 1 = |x| < 1.
\end{aligned}$$

   By the ratio test, (a_n) converges.

2. As we have determined in (a), $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ converges for $|x| < 1$. We can differentiate power series term-by-term within their radii of convergence, so for $|x| < 1$,

$$\begin{aligned}
(1+x)f'(x) &= (1+x) \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1} \\
&= \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1} + \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^n \\
&= \sum_{n=0}^{\infty} (n+1) \binom{\alpha}{n+1} x^n + \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^n
\end{aligned}$$

$$
\begin{aligned}
&= (0+1)\binom{\alpha}{0+1}x^0 + \sum_{n=1}^{\infty}\left[(n+1)\binom{\alpha}{n+1} + n\binom{\alpha}{n}\right]x^n \\
&= \alpha + \sum_{n=1}^{\infty}\left[(n+1)\frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!} + n\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}\right]x^n \\
&= \alpha + \sum_{n=1}^{\infty}\left[\frac{(\alpha-n)[\alpha(\alpha-1)\dots(\alpha-n+1)] + n[\alpha(\alpha-1)\dots(\alpha-n+1)]}{n!}\right]x^n \\
&= \alpha + \sum_{n=1}^{\infty}\left[\frac{\alpha[\alpha(\alpha-1)\dots(\alpha-n+1)]}{n!}\right]x^n \\
&= \alpha + \alpha\sum_{n=1}^{\infty}\binom{\alpha}{n}x^n \\
&= \alpha\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n = \alpha f(x).
\end{aligned}
$$

3. Since f is differentiable and $(1+x)^\alpha \neq 0$ on $|x|<1$, $g(x) = f(x)/(1+x)^\alpha$ is differentiable for $|x|<1$. We have

$$
\begin{aligned}
g'(x) &= \frac{f'(x)(1+x)^\alpha - \alpha f(x)(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} \\
&= \frac{(1+x)^{\alpha-1}((1+x)f'(x) - \alpha f(x))}{(1+x)^{2\alpha}} \\
&= 0
\end{aligned}
$$

by assumption that $(1+x)f'(x) = \alpha f(x)$. Thus $g(x) = c$ for some constant c, and

$$f(x) = g(x)(1+x)^\alpha = c(1+x)^\alpha.$$

4. Let $f(x) = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$ again. From (c), we know that $f(x) = c(1+x)^\alpha$ for some c. At $x=0$, we have

$$1 = \binom{\alpha}{0} = f(0) = c(1+0)^\alpha = c.$$

This completes the proof.

Problem 2
Suppose that $(f_n)_{n\in\mathbb{N}} : [a,b] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly to f. Show that if $x_n \to x$ where $x_n \in [a,b]$, then $f_n(x_n) \to f(x)$. Is this true if we don't assume the f_n are continuous? Is it true if the convergence is not uniform?

Solution

We want to show the sequence of real numbers $(f_n(x_n))$ converges to $f(x)$. Let $\epsilon > 0$. Since $(f_n) \to f$ uniformly, choose $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|f_n(y) - f(y)| < \epsilon/4$ for all $y \in [a,b]$. Notice that this condition guarantees that if $n \geq N_1$, $|f_n(y) - f_{N_1}(y)| < \epsilon/2$ for all y, since

$$|f_n(y) - f_{N_1}(y)| \leq |f_n(y) - f(y)| + |f_{N_1}(y) - f(y)| < \epsilon/4 + \epsilon/4 = \epsilon/2. \tag{2}$$

Since f_{N_1} is continuous (at x in particular), there is some $\delta > 0$ such that if $|x-y| < \delta$ then $|f_{N_1}(x) - f_{N_1}(y)| < \epsilon/2$. Choose $N_2 \in \mathbb{N}$ such that $|x_n - x| < \delta$ for $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $|x_n - x| < \delta$, so $|f_{N_1}(x_n) - f_{N_1}(x)| < \epsilon/2$. Thus

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_{N_1}(x_n)| + |f_{N_1}(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

where the inequality $|f_n(x_n) - f_{N_1}(x_n)| < \epsilon/2$ is due to (2).

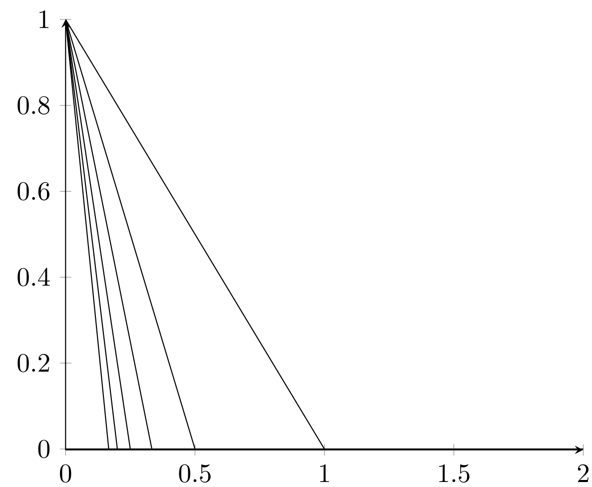
The statement is not true if we don't assume the f_n are continuous. Take

$$f_n(x) = f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

over $[0,1]$ for example. $f_n \to f$ uniformly, but if $x_n = \frac{1}{n}$, $x_n \to 0$ but $f_n(x_n) \not\to f(0)$.

The statement is also not true if we only assume $f_n \to f$ pointwise only. Define, over $[0,2]$,

$$f_n(x) = \begin{cases} 1 - nx & x \le \frac{1}{n} \\ 0 & x > \frac{1}{n}. \end{cases}$$



$f_n \to f$ pointwise, where

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0. \end{cases}$$

Setting $x_n = \frac{2}{n}$, we have $x_n \to 0$. But $f_n(x_n) = 0$ for all n, while $f(0) = 1$.

Problem 3

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even function. Show that $a_n = 0$ for every odd $n \in \mathbb{N}$. If f is odd instead, show that $a_n = 0$ for every even $n \in \mathbb{N}$.

Solution

Since f is even, $f(x) - f(-x) = 0$ for all x. So

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} (-1)^n a_n x^n = 0.$$

Adding the two series together, the even terms cancel out, while the odd terms add:

$$\sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} 2a_n x^n = 0.$$

A power series is identically zero if and only if all of its coefficients are zero, so the above shows that a_n must be zero for all odd n.

If f is odd instead, $f(x) + f(-x) = 0$ for all x, and the odd terms would cancel out:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} (-1)^n a_n x^n = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} a_n x^n = 0.$$