Tutorial 3 (some test review probably)

Problem 1

Determine if the *maps* below are valid *functions*.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$.
2. $f : \mathbb{R} \to [0, 2\pi], f(x) = \cos\left(\frac{1}{x^2}\right)$.
3. $f : (0, 1) \to \mathbb{N}, f(x) = 2^{x_1}3^{x_2}5^{x_3}7^{x_4}\ldots$, where $0.x_1x_2x_3x_4\ldots$ is the decimal representation of x.
4. $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 2x + 1 & x \geq 0 \\ 1 - x^2 & x \leq 0 \end{cases}$.

1. $f: \mathbb{R} \to \mathbb{R}$ domain is $\mathbb{R}$, but $\sqrt{-\text{ve number}}$ is not defined

2. $f: \mathbb{R} \to [0, 2\pi]$ domain is $\mathbb{R}$, but $\cos\left(\frac{1}{0^2}\right)$ not defined.

Even if the domain was $\mathbb{R} \setminus \{0\}$, we're not guaranteed to output something in $[0, 2\pi]$

3. $f: (0,1) \to \mathbb{N}$

What is $f(0.1111\ldots)$?

$f(0.1111\ldots) = \underbrace{2^1 3^1 5^1 7^1 \ldots}_{\text{infinite product, not defined.}}$

$f(0.1000\ldots) = 2^1 3^0 5^0 7^0 \ldots$

$f(0.0999\ldots) = 2^9 3^9 5^9 7^9 \ldots$

4. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ 1-x^2 & x \leq 0 \end{cases}$$

if $x = 0$, is $f(x) = 2x+1$ or $f(x) = 1 - x^2$

it doesn't matter!

if $x = 0$, $2x+1 = 1$

$1 - x^2 = 1$

so either way, $f(0) = 1$.

So f is indeed a well-defined function.

Problem 2

Let $f : B \to C$, $g : A \to B$ ~~be surjective.~~

1. What is the domain and codomain of $f \circ g$?
2. Show that $f \circ g$ is surjective.
3. Suppose, instead of knowing that f and g are both surjective, that we only know $f \circ g$ is surjective. Must f be surjective? Must g be surjective?

1. $f \circ g : A \to C$

$$\underbrace{A \xrightarrow{g} B \xrightarrow{f} C}_{f \circ g}$$

2.

To show $f \circ g : A \to C$ is surjective, we must prove $(\forall c \in C)(\exists a \in A)\ (f \circ g)(a) = c$.

We know: $f : B \to C$, $g : A \to B$ surjective.

Let $c \in C$. Want to find $a \in A$ s.t. $(f \circ g)(a) = c$.

Since f surjective, $\exists\, b \in B$ s.t. $f(b) = c$.

Since g surjective, $\exists\, a \in A$ s.t. $g(a) = b$.

Then $(f \circ g)(a) = f(g(a)) = f(b) = c$.

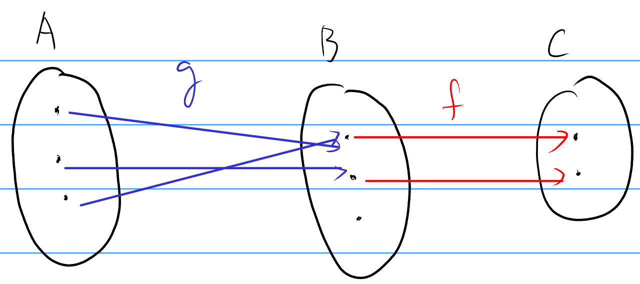
3. Yes, we can conclude f surjective.

$(\forall c \in C)(\exists a \in A)\ (f \circ g)(a) = c$.

Then, for any $c \in C$, taking $b = g(a)$, we have $f(b) = c$.

So f surjective.

g might not be surjective!



Problem 3

1. State the *completeness axiom*.
2. Show that the completeness axiom doesn't hold if $\mathbb{R}$ is replaced with $\mathbb{Q}$.

1. For any $S \subseteq \mathbb{R}$, $S \neq \emptyset$, such that S is bounded from above, there is a least upper bound for S in $\mathbb{R}$.

2. $S = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$. ← set of rationals $< \sqrt{2}$.

$\cancel{\sqrt{2}}$ ← not a valid upper bound in $\mathbb{Q}$!

Note: any rational upper bound for S must be at least $\sqrt{2}$.

(if $x < \sqrt{2}$, $x \in \mathbb{Q}$, then we can find a rational between x and $\sqrt{2}$)

But any rational upper bound $x \geq \sqrt{2}$ must have an even lower rational upper bound between $\sqrt{2}$ and x.

Problem 4
Let $S \subseteq \mathbb{R}$. Give two equivalent definitions for "$M = \sup S$". Give two equivalent definitions for "$m = \inf S$".

$M = \sup S \Leftrightarrow$ M is an upper bound for S, and any upper bound x for S has $x \geq M$.

$\Leftrightarrow (\forall \varepsilon > 0)(\exists s \in S)\,[M - \varepsilon < s \leq M]$ and M is an upper bound for S.

$m = \inf S \Leftrightarrow$ m is a lower bound for S, and any lower bound x for S has $x \leq m$.

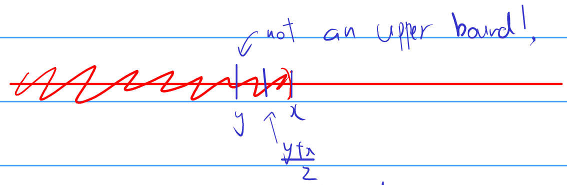
$\Leftrightarrow (\forall \varepsilon > 0)(\exists s \in S)\,[m \leq s < m + \varepsilon]$ and m is a lower bound for S.

Problem 5
Show that $\sup(-\infty, x) = x$ for any $x \in \mathbb{R}$.

① Using the least upper bound definition

- x is an upper bound for $(-\infty, x)$.
  all real numbers $< x$.
- Let y be an upper bound for $(-\infty, x)$. WTS $y \geq x$.

for the sake of contradiction → FTSOC, assume $y < x$.



Then $\frac{y+x}{2} \in (-\infty, x)$

$\frac{y+x}{2} < \frac{x+x}{2} = x$.

but $\frac{y+x}{2} > y$. So y is not an upper bound for $(-\infty, x)$!

Contradiction.

Thus $y \geq x$ as needed.

② Using the ε definition.

WTS $(\forall \varepsilon > 0)(\exists s \in (-\infty, x))\,[x - \varepsilon < s \leq x]$.

And x is an upper bound for $(-\infty, x)$. ✓

Let $\varepsilon > 0$.

$\frac{x + x - \varepsilon}{2} = x - \frac{\varepsilon}{2}$.

Choose $s = x - \frac{\varepsilon}{2}$. $s < x$ so $s \in (-\infty, x)$.

$x - \varepsilon < x - \frac{\varepsilon}{2} \Rightarrow x - \varepsilon < s$.

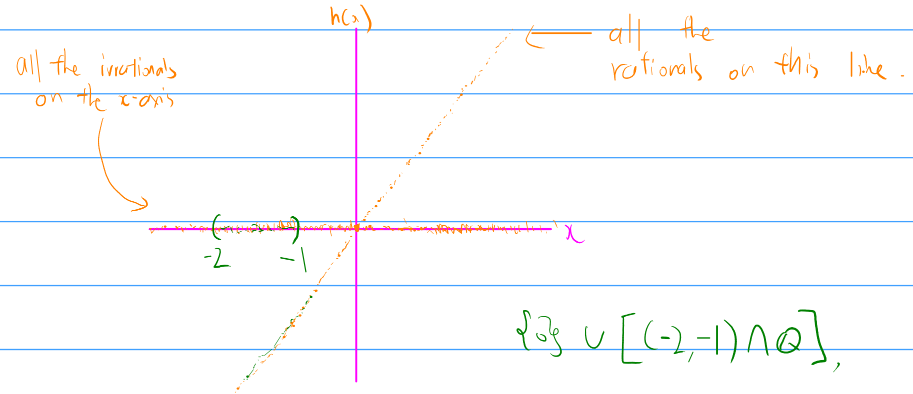
So $x - \varepsilon < s \leq x$. $\square$

Density of Rationals

Problem 6
Let

$$h : \mathbb{R} \to \mathbb{R}, h(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x & x \in \mathbb{Q}. \end{cases}$$

Show that $\sup\{h(x) : x \in (a, b)\} = b$, for any open interval $(a, b) \subseteq \mathbb{R}, a < b$, $b > 0$.



Using ε definition:

① Show b is an upper bound for $\{h(x) : x \in (a,b)\}$

② Show $(\forall \varepsilon > 0)(\exists s \in \{h(x) : x \in (a,b)\})\ b - \varepsilon < s \leq b$.

For ①, Let $s \in \{h(x) : x \in (a,b)\}$. So $s = h(x)$ for some $x \in (a,b)$.

if $x \in \mathbb{Q}$, then $s = h(x) = x$. Since $x \in (a,b)$, $s < b$.

if $x \notin \mathbb{Q}$, then $s = h(x) = 0$. Since $b > 0$, $s < b$.

For ②, Let $\varepsilon > 0$. If $b - \varepsilon < 0$, then taking an irrational

$x \in (a,b) \setminus \mathbb{Q}$, $b - \varepsilon < 0 = h(x) < b$.

So if $s = h(x)$,

$b - \varepsilon < s \leq b$.

If $b - \varepsilon > 0$, choose a

$x \in (b - \varepsilon, b) \cap \mathbb{Q}$.

$b - \varepsilon < h(x) = x < b$.

Thus if $s = h(x)$,

$b - \varepsilon < s \leq b$.

Problem 7

In this question, we provide a proof that the *irrationals* are *dense*.

1. Define what it means for a set $S \subseteq \mathbb{R}$ to be dense.
2. Define *countable* and *uncountable* sets. Recall that $\mathbb{Q}$ is countable, while $\mathbb{R}$ is uncountable.
3. Show that $|\mathbb{R}| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$ by defining a bijection between them. If you prefer, you may draw a graph instead of explicitly defining this bijection. This shows $(-\frac{\pi}{2}, \frac{\pi}{2})$ is uncountable.
4. Show that $|(a,b)| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$ for any $a < b$. This shows any open interval (a,b) is uncountable.
5. Prove by contradiction that any open interval (a,b) contains an irrational number. Conclude that the irrationals are dense.

1. $S \subseteq \mathbb{R}$ is <u>dense</u> when $(a,b) \cap S \neq \emptyset$ for any open interval (a,b), $a<b$

$\mathbb{N} \cap (\frac{1}{2}, \frac{3}{4}) = \emptyset$

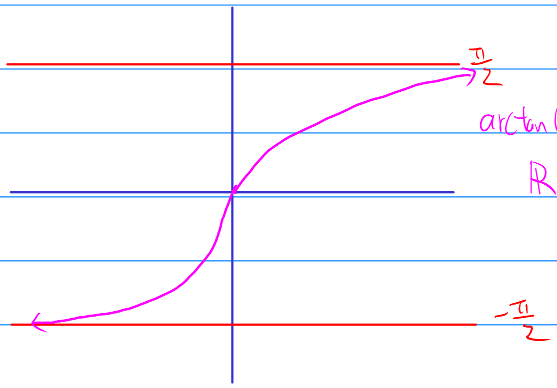
$\mathbb{N}$ is not dense.

2. A set S is <u>countable</u> when $|S| \leq |\mathbb{N}|$

there is an injection $f: S \to \mathbb{N}$.

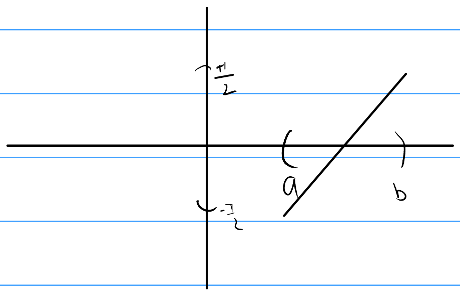
A set S is uncountable when it's not countable.

3.



arctan(x) is a bijection from $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

4. $|(a,b)| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$



bijection between (a,b) and $(-\frac{\pi}{2}, \frac{\pi}{2})$

(5) any open interval (a,b) is uncountable.

if some open interval (a,b) didn't contain any irrationals,

then $(a,b) = (a,b) \cap \mathbb{Q}$.

cardinality $\leq |\mathbb{Q}|$, so (a,b) countable!

Thus, every open interval (a,b) contains an irrational.