Problem 1
Let $f(x) = 3x - 4$. Let $\epsilon = \frac{1}{100}$. Find a $\delta > 0$ so that for all $x \in \mathbb{R}$,

$$0 < |x - 2| < \delta \implies |f(x) - 2| < \frac{1}{100}$$

Solution
If $\delta = \frac{1}{300}$, then for any x, if $0 < |x - 2| < \delta$, then

$$|f(x) - 2| = |(3x - 4) - 2| = |3x - 6| = 3|x - 2| < \frac{3}{300} = \frac{1}{100}.$$

Problem 2
Show that $\lim_{x \to 0} \frac{1}{x} \neq 100$ by proving the negation:

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}) \left[0 < |x| < \delta \wedge \left|\frac{1}{x} - 100\right| \geq \epsilon\right]$$

Hint: since we are allowed to choose ϵ and x, it suffices to let $\epsilon = 1$ and only consider positive values of x.

Solution Choose $\epsilon = 1$. Let $\delta > 0$ be arbitrary. We consider two cases:

- If $\delta > \frac{1}{101}$, then choosing $x = \frac{1}{101}$, we have $0 < |x| < \delta$ and
$$\left|\frac{1}{x} - 100\right| = \left|\frac{1}{\frac{1}{101}} - 100\right| = 1 \geq \epsilon.$$
- If $\delta < \frac{1}{101}$, then choosing $x = \frac{\delta}{2}$, we have $0 < |x| < \delta$, and since $\frac{1}{x} > \frac{1}{\delta} = 101$,
$$\left|\frac{1}{x} - 100\right| = \frac{1}{x} - 100 > 101 - 100 = \epsilon.$$

Problem 3
For each of the following problems you may draw a graph to support your reasoning instead of giving a full proof.[a]

1. Define $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Does $\lim_{x \to 0} h(x)$ exist?
2. Define $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Does $\lim_{x \to 0} h(x)$ exist?
3. Define $h : (-1, 1) \setminus \{0\} \to \mathbb{R}$, $h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Does $\lim_{x \to 0} h(x)$ exist?

[a] Don't do this on your assignment.

Solution It's really inconvenient to draw a graph here, so I'll just give a full proof.

1. No. We prove that $\lim_{x \to 0} h(x) \neq L$ for any $L \in \mathbb{R}$, i.e.
$$(\forall L \in \mathbb{R})(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(0 < |x| < \delta \wedge |h(x) - L| \geq \epsilon).$$
Let $L \in \mathbb{R}$. We consider two cases:

- If $L = 0$, then choose $\epsilon = 1$. Notice that whenever $x \in \mathbb{Q}$, $|h(x) - L| = 1$. So for any $\delta > 0$, choosing a rational $x \in (-\delta, \delta) \setminus \{0\}$, then $0 < |x| < \delta$ and $|h(x) - L| = 1 \geq \epsilon$.
- If $L \neq 0$, choose $\epsilon = |L| > 0$. Let $\delta > 0$. We may choose an irrational $x \in (-\delta, \delta)$, so that $0 < |x| < \delta$ and $|h(x) - L| = |0 - L| = |L| \geq \epsilon$.

Thus the limit doesn't exist.

2. The limit exists and is equal to 0. To prove this, let $\epsilon > 0$. We choose $\delta = \epsilon$. Then for any x such that $0 < |x| < \delta$,
   - If $x \in \mathbb{Q}$, then $|h(x) - 0| = |x| < \delta = \epsilon$.
   - If $x \notin \mathbb{Q}$, then $|h(x) - 0| = 0 < \epsilon$.
3. The limit exists here for the same reason it exists in (2). Notice that when we are evaluating $\lim\limits_{x\to 0} h(x)$, h doesn't actually have to be defined at 0.

Problem 4
Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{x\to\infty} f(x) = L.$$

1. Write down the definition of
$$\lim_{x\to\infty} f(x) = L.$$
2. Write down the definition of
$$\lim_{y\to 0^+} f\left(\frac{1}{y}\right) = L.$$
3. Show that
$$\lim_{y\to 0^+} f\left(\frac{1}{y}\right) = L.$$

Solution

1. $(\forall \epsilon > 0)(\exists N \in \mathbb{R})(\forall x \in \mathbb{R})[x > N \Rightarrow |f(x) - L| < \epsilon]$.
2. $(\forall \epsilon > 0)(\exists \delta \in \mathbb{R})(\forall y \in \mathbb{R})[0 < y < \delta \Rightarrow \left|f\left(\frac{1}{y}\right) - L\right| < \epsilon]$.
3. We show that the statement in the previous subproblem holds. Let $\epsilon > 0$.

   Since $\lim\limits_{x\to\infty} f(x) = L$, we may find a $N \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x > N \Rightarrow |f(x) - L| < \epsilon$. We can assume that $N > 0$, since if $N \leq 0$ we can just choose a positive N instead. Now since $N > 0$, if we let $\delta = \frac{1}{N}$, then $\delta > 0$.

   Let $y \in \mathbb{R}$, and suppose $0 < y < \delta$. Then $\frac{1}{y} > \frac{1}{\delta} = N$, so by our choice of N, $\left|f\left(\frac{1}{y}\right) - L\right| < \epsilon$.

Problem 5
Recall that if f, g are defined in some interval around $c \in \mathbb{R}$, and

$$\lim_{x\to c} f(x) = M \quad \text{and} \quad \lim_{x\to c} g(x) = N,$$

then

$$\lim_{x\to c}[f(x) + g(x)] = M + N \quad \text{and} \quad \lim_{x\to c}[f(x)g(x)] = MN.$$

This problem shows why it is necessary for $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ to exist in the above.

1. Give an example of an f, g defined in an interval around $c \in \mathbb{R}$ such that $\lim\limits_{x\to c}[f(x) + g(x)]$ exists

but $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ don't exist.

2. Give an example of an f, g defined in an interval around $c \in \mathbb{R}$ such that $\lim_{x\to c}[f(x)g(x)]$ exists but $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ don't exist.

Solution

1. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$, and $g : \mathbb{R} \to \mathbb{R}, g(x) = -f(x)$. Let $c = 0$. As we have shown in Problem 3, $\lim_{x\to c} f(x)$ doesn't exist. But notice that for any $x \in \mathbb{R}$,
$$f(x) + g(x) = 0.$$
Thus $f(x) + g(x)$ is identically zero. So
$$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c}[0] = 0.$$

2. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$, and $g : \mathbb{R} \to \mathbb{R}, g(x) = 0$. Let $c = 0$. Again, $\lim_{x\to c} f(x)$ doesn't exist, and for all $x \in \mathbb{R}$,
$$f(x)g(x) = 0.$$
So
$$\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c}[0] = 0.$$

Problem 6

Use squeeze theorem in the following questions.

1. Write down the statement of squeeze theorem.
2. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$
Show that $\lim_{x\to 0} f(x) = 0$.
3. Show that for any function $f : \mathbb{R} \to \mathbb{R}$,
$$\lim_{x\to c} |f(x)| = 0$$
if and only if
$$\lim_{x\to c} f(x) = 0.$$

1. Suppose $f, g, h : I \to \mathbb{R}$, with I an interval, satisfy for all x,
$$f(x) \leq g(x) \leq h(x)$$
and
$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x).$$
Then $\lim_{x\to c} g(x)$ exists and
$$\lim_{x\to c} g(x) = L.$$

2. Define $f_1 : \mathbb{R} \to \mathbb{R}, f_1(x) = 0$, and $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = |x|$. We have, for all x,

$$f_1(x) \leq f(x) \leq f_2(x).$$

Furthermore,[1]

$$\lim_{x\to 0} f_1(x) = 0 = \lim_{x\to 0} f_2(x).$$

Thus by squeeze theorem,

$$\lim_{x\to 0} f(x) = 0.$$

3. Suppose $\lim_{x\to c} |f(x)| = 0$. Notice that for all x,

$$-|f(x)| < f(x) < |f(x)|.$$

We have

$$\lim_{x\to c} |f(x)| = 0$$

and through the limit laws,

$$\lim_{x\to c}[-|f(x)|] = -\lim_{x\to c} |f(x)| = 0.$$

Since the two limits are equal, we may apply squeeze theorem to conclude that

$$\lim_{x\to c} f(x) = 0.$$

As for the converse, actually I realized that squeeze theorem isn't the best way to prove this, as the argument is more convoluted than just using ϵ-δ, and it involves an ϵ-δ argument in proving some other limit. But here it is anyway.

Suppose $\lim\limits_{x\to c} f(x) = 0$. We have, for all x,

$$\min\{-f(x), f(x)\} \leq |f(x)| \leq \max\{-f(x), f(x)\}.$$

Notice that

$$\lim_{x\to c} \min\{-f(x), f(x)\} = 0.$$

I'll leave the full argument for this to you, but if $|f(x)| < \epsilon$, then $|-f(x)| < \epsilon$ as well. Since $\min\{-f(x), f(x)\}$ is equal to either $-f(x)$ or $f(x)$, we have $|\min\{-f(x), f(x)\}| < \epsilon$.

Similarly,

$$\lim_{x\to c} \max\{-f(x), f(x)\} = 0.$$

So applying squeeze theorem $\lim_{x\to c} |f(x)| = 0$.

[1] I haven't justified why

$$\lim_{x\to 0} f_2(x) = 0$$

but it shouldn't be too much work to do this with an ϵ-δ argument. I'll leave it up to you to complete this proof, if you'd like.