MAT157 Tutorial 5

A function $f : I \to \mathbb{R}$ is **continuous at** c when

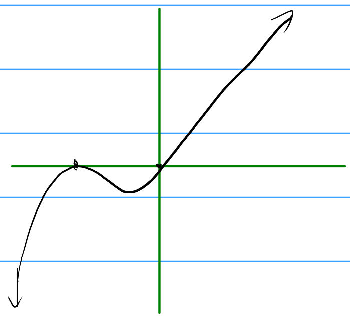
$$\lim_{x \to c} f(x) = f(c).$$

If f is continuous at every $c \in I$, we simply say f is **continuous**.

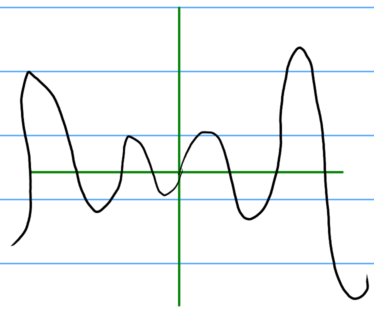
Problem 1
Show that each of the following statements is false using a counterexample.

1. A continuous, surjective function $f : \mathbb{R} \to \mathbb{R}$ is injective.
2. A continuous, injective function $f : \mathbb{R} \to \mathbb{R}$ is surjective.
3. A bijective function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
4. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f \circ g$ is continuous, then f is continuous.
5. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f \circ g$ is continuous, then g is continuous.
6. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are such that $f \circ g$ is continuous, then either f or g is continuous.

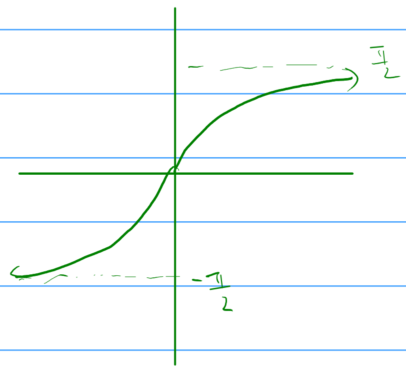
1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$



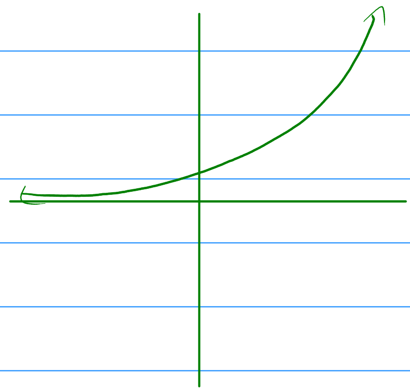
$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x \sin(x)$

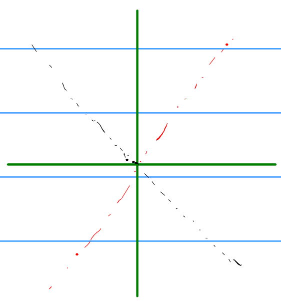


2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \arctan(x)$



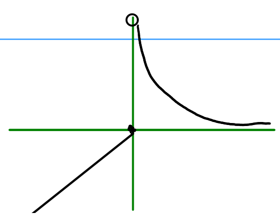
$f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$



3. 

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q}. \end{cases}$

Exercise: show f bijective.



$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x & x \leq 0 \\ \frac{1}{x} & x > 0 \end{cases}$

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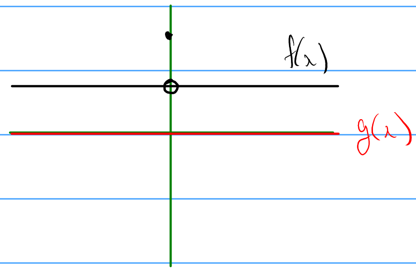
6 min 10:32.

4. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$

$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = 0$

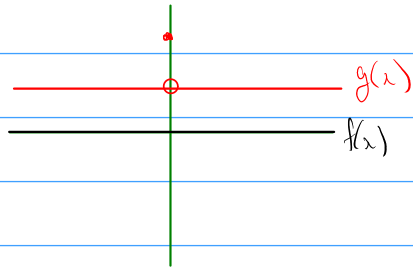
$(f \circ g)(x) = 2$ for all x.

$f \circ g$ continuous.

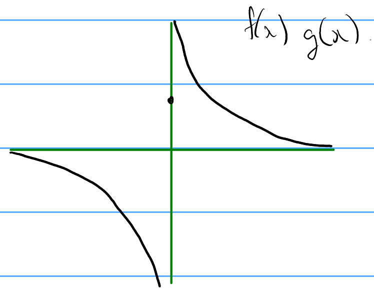


5. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 0$

$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$



6. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$

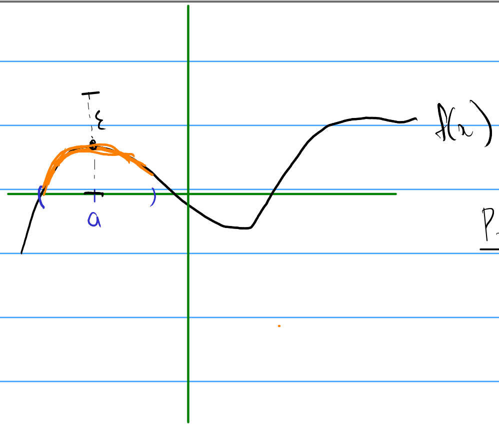


$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$

$f(g(x)) = x \quad \forall x \in \mathbb{R}$

$f \circ g$ continuous

Problem 2
Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ and $f(a) > 0$, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (a - \delta, a + \delta)$.



Pf since f is continuous at a,

$(\forall \varepsilon > 0)(\exists \delta > 0)(\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$

Let $\varepsilon = f(a) > 0$.

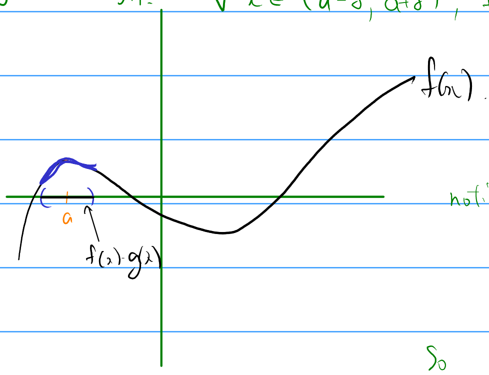
Then $\exists\, \delta > 0$ s.t.

$(\forall x \in (a-\delta, a+\delta))\big(f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)\big)$
$\qquad\qquad = (0, 2f(a))$

Thus, for all $x \in (a-\delta, a+\delta)$,
we have $f(x) > 0$.

Problem 3
Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, and we have for some $a \in \mathbb{R}$, $f(a) > 0$ and $(fg)(\cancel{a}\ x) = 0$ $\forall x \in \mathbb{R}$, there exists some $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$, $g(x) = 0$. *Hint: Use the previous problem.*

Pf Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous, $f(a) > 0$, and $(fg)(x) = 0$, $\forall x \in \mathbb{R}$.
Using the previous problem, since f continuous, and $f(a) > 0$,
$\exists\, \delta > 0$ s.t. $\forall x \in (a-\delta, a+\delta)$, $f(x) > 0$.



notice $f(x)\cdot g(x) = 0 \quad \forall x \in (a-\delta, a+\delta)$

but $f(x) > 0$ on $(a-\delta, a+\delta)$.

So for any $x \in (a-\delta, a+\delta)$,

$0 = f(x) \cdot g(x) = (\text{something} \neq 0) \cdot g(x)$

So $g(x) = 0$. $\forall x \in (a-\delta, a+\delta)$.

Problem 4

Show that if $f : \mathbb{R} \to \mathbb{R}$, and $(a, b) \subseteq \mathbb{R}$ is an interval, then f is continuous if and only if for any $\underbrace{x \in f^{-1}((a, b))}_{\{x \in \mathbb{R} : f(x) \in (a,b)\}}$ there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq f^{-1}((a, b))$.

"preimage of (a,b)"

$x \in f^{-1}((a,b)) \Leftrightarrow f(x) \in (a,b)$.

$(\Rightarrow)$ 11:23

Suppose f continuous. Let $(a,b) \subseteq \mathbb{R}$ be an interval.

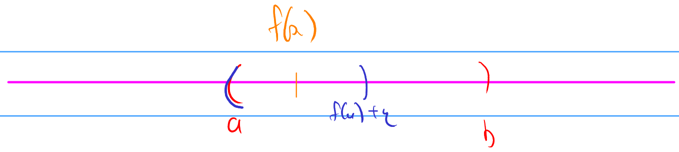
Suppose $x \in f^{-1}((a,b))$.

Want to find $\delta > 0$ s.t. $(x-\delta, x+\delta) \subseteq f^{-1}((a,b))$.

$\forall y \in (x-\delta, x+\delta)$, $f(y) \in (a,b)$.

Since f continuous at x,

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall y \in \mathbb{R})\ (|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon).$$



Let $\varepsilon = \min\{|f(x) - a|, |f(x) - b|\} > 0$.

Notice $(f(x) - \varepsilon, f(x) + \varepsilon) \subseteq (a,b)$.

Chose δ accordingly so that

$$(\forall y \in \mathbb{R})(\underbrace{|y - x| < \delta}_{y \in (x-\delta, x+\delta)} \Rightarrow \underbrace{|f(y) - f(x)| < \varepsilon}_{f(y) \in (f(x)-\varepsilon,\ f(x)+\varepsilon) \subseteq (a,b)})$$

$$y \in (x-\delta, x+\delta) \Rightarrow y \in f^{-1}((a,b))$$

$(\Leftarrow)$ Suppose $\forall x \in f^{-1}((a,b))$, $\exists \delta > 0$ s.t. $(x-\delta, x+\delta) \subseteq f^{-1}((a,b))$.

We show f is continuous at c for all $a \in \mathbb{R}$:

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon).$$

Let $\varepsilon > 0$. $f(c) \in (f(c) - \varepsilon, f(c) + \varepsilon)$

$\Rightarrow c \in f^{-1}((f(c) - \varepsilon, f(c) + \varepsilon))$

From our assumption, $\exists\ \delta > 0$ s.t.

$$(c-\delta, c+\delta) \subseteq f^{-1}\big((f(c)-\varepsilon, f(c)+\varepsilon)\big)$$

So for all $x \in (c-\delta, c+\delta)$,

$$x \in f^{-1}\big((f(c)-\varepsilon, f(c)+\varepsilon)\big)$$

$$\Rightarrow f(x) \in (f(c)-\varepsilon, f(c)+\varepsilon)$$

$$\Rightarrow |f(x) - f(c)| < \varepsilon.$$

Thus, with the same δ,

$\forall x \in \mathbb{R}$ if $|x-c| < \delta$

then $|f(x) - f(c)| < \varepsilon$.

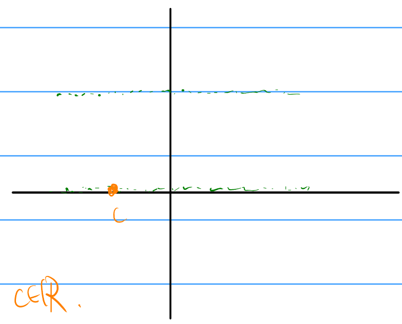
Problem 5
Exhibit a function $f : I \to \mathbb{R}$ which is:

1. Everywhere discontinuous.
2. Continuous only at 0.
3. Continuous only at integers.
4. Continuous only at irrational numbers.

1. We want to find $f : \mathbb{R} \to \mathbb{R}$ s.t.

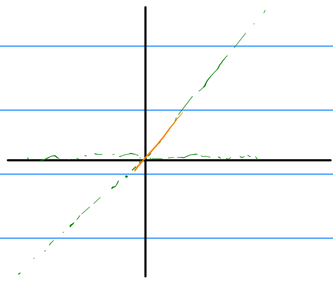
$$\lim_{x\to c} f(x) \neq f(c) \quad \forall c \in \mathbb{R}.$$

Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$



$\lim_{x\to c} f(x)$ DNE for all $c \in \mathbb{R}$.

2. Let $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

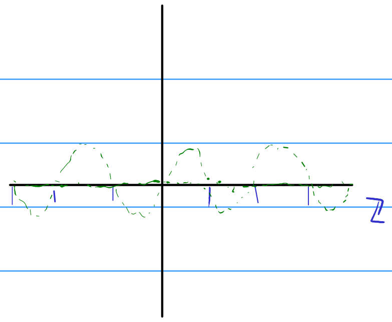


Exercise: $\lim_{x\to c} f(x)$ DNE

for $c \neq 0$,

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

3. $f(x) = \begin{cases} \sin(\pi x) & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$



$\lim_{x \to c} f(x)$ DNE for $c \notin \mathbb{Z}$

$\lim_{x \to c} f(x) = 0 = f(c)$ for $c \in \mathbb{Z}$.

4. Thomae's function.

$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$

if $q > \frac{1}{\varepsilon}$, then $\frac{1}{q} < \varepsilon$.

$\lim_{x \to c} f(x) = 0$ if c irrational

Pf Let $c \notin \mathbb{Q}$. Let $\varepsilon > 0$. Chose $q \in \mathbb{N}$ large enough so that $\frac{1}{q} < \varepsilon$.

Chose $\delta_1 > 0$ so that $(c - \delta_1, c + \delta_1)$ does not contain any number of the form $\frac{p}{1}$, $p \in \mathbb{Z}$

Choose $\delta_2 > 0$ so that $(c - \delta_2, c + \delta_2)$ does not contain any number of the form $\frac{p}{2}$, $p \in \mathbb{Z}$.

$\vdots$

Chose $\delta_{q-1} > 0$ so that $(c - \delta_{q-1}, c + \delta_{q-1})$ does not contain any number of the form $\frac{p}{q-1}$, $p \in \mathbb{Z}$.

Let $\delta = \underbrace{\min\{\delta_1, \delta_2, \ldots, \delta_{q-1}\}}_{\text{min exists because } \{\delta_1, \delta_2, \ldots \delta_{q-1}\} \text{ finite.}} > 0$.

Then $(c - \delta, c + \delta) \subseteq (c - \delta_i, c + \delta_i)$ for $i = 1, 2, \ldots, q-1$,

because $\delta \leq \delta_i$.

So $(c - \delta, c + \delta)$ does not contain any numbers

$\frac{p}{1}, \frac{p}{2}, \ldots, \frac{p}{q-1}$, Thus, $(c-\delta, c+\delta)$ only contains

- rationals $\frac{p}{d}$, where it's in lowest terms, and $d \geq q$.
- irrationals.

So for all $x \in (c-\delta, c+\delta)$:

- if x is rational, then write $x = \frac{p}{d}$, where it's in lowest terms, and $d \geq q$.
  $f(x) = \frac{1}{d} \leq \frac{1}{q} < \varepsilon$.
- if $x \notin \mathbb{Q}$, then $f(x) = 0$.

In either case, $|f(x) - f(c)| < \varepsilon$.

So $(|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon)$.

Exercise: $\lim_{x \to c} f(x) = 0 \neq f(c)$ if c is rational.