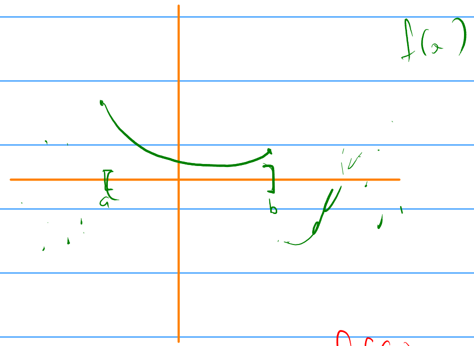
MAT157 Tutorial 6

Firstly, some more practice on continuous functions.

Problem 1

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$. Construct a function g which is continuous on $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all $x \in [a, b]$.
2. Give an example to show that this doesn't need to be true if we only assume f is continuous on (a, b).



a) $g(x) = \begin{cases} f(a) & x < a \\ f(x) & a \le x \le b \\ f(b) & x > b \end{cases}$

$g(x) = f(x) \quad \forall x \in [a,b]$ by def. of $g(x)$.

We show $g(x)$ cts $\forall c \in \mathbb{R}$.

Case 1: $c < a$ (so $g(c) = g(a)$)

$\forall \varepsilon > 0$, Take $\delta = a - c$.

Then $|x-c| < \delta \Rightarrow -(a-c) < \underbrace{x - c < a - c}$

$\Rightarrow x < a$.

$\Rightarrow g(x) = f(a)$

$|g(x) - g(c)| = |g(a) - g(a)| = 0 < \varepsilon$.

Case 2: $c > b$ similar.

Case 3: $c = a$: Since f is right cts at a

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in [a, a+\delta) \ |f(x) - f(a)| < \varepsilon$.

Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $\forall x \in [a, a+\delta) \ |f(x) - f(a)| < \varepsilon$.

The same δ gives $\forall x \in (a-\delta, a+\delta) \ |g(x) - g(a)| < \varepsilon$

$g(x) \quad g(a)$

Cuz if $x \in [a, a+\delta) \Rightarrow |f(x) - f(a)| < \varepsilon$

$x \in (a-\delta, a) \Rightarrow g(x) = g(a) = f(a)$

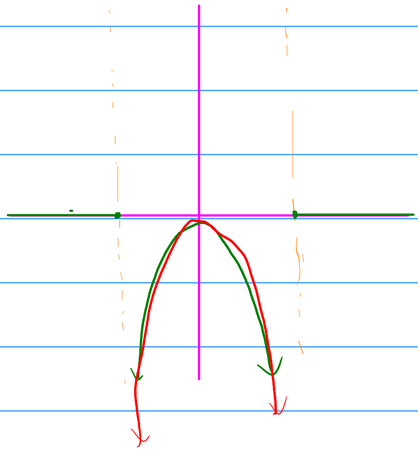
$|g(x) - g(a)| = 0 < \varepsilon$.

Case 4: $c = b$ similar

Case 5: $c \in (a,b)$. f cts at c, so g cts at c.

b)

2. Give an example to show that this doesn't need to be true if we only assume f is continuous on (a, b).



$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{1}{x^2-1} = \begin{cases} \frac{1}{(x+1)(x-1)} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

no way to define $g(x)$ such that it's continuous at a or b.

Problem 2
Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Furthermore, suppose f is continuous at 0. Show that f is continuous everywhere.
Hint: Show that $f(0) = 0$ first.

Hint 2: Show that $f(-x) = -f(x)$ for all x.

① Show $f(0) = 0$.

Pf $f(0) = f(0+0) = f(0) + f(0)$

$0 = f(0)$.

② Show $f(-x) = -f(x)$ $\forall x \in \mathbb{R}$.

Pf $f(x + (-x)) = f(0) = 0$

$\quad \| $

$f(x) + f(-x)$

$f(-x) = -f(x)$

③ Show f cts at c $\forall c \in \mathbb{R}$.

Pf Let $\varepsilon > 0$.

We know f cts at 0, so

$\exists \delta > 0$ $\forall x \in \mathbb{R}$ $|x| < \delta \Rightarrow |f(x) - f(0)| = |f(x)| < \varepsilon$.

Same delta works to show f cts at c:

if $|x - c| < \delta$, then

$|f(x-c)| < \varepsilon$ (by choice of δ)

$\quad \| $

$|f(x) + f(-c)| \overset{②}{=} |f(x) - f(c)|$.

A function $f : D \to \mathbb{R}$ is **uniformly continuous** when

$$(\forall \epsilon > 0)(\exists \delta)(\forall x, y \in D)[|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon].$$

Problem 3

1. Describe the (subtle but important) difference between "uniformly continuous" and "continuous".
2. Briefly explain why uniform continuity implies continuity. In other words, show if $f : D \to \mathbb{R}$ is uniformly continuous, then it is continuous everywhere in D.
3. Give an example of a function $f : D \to \mathbb{R}$ that is continuous but not uniformly continuous.

1. f Uniformly cts

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in D)\,[|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon].$$

f cts $\Leftrightarrow$ f cts at all $c \in D$

$$(\forall c \in D)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D)\,[|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon].$$

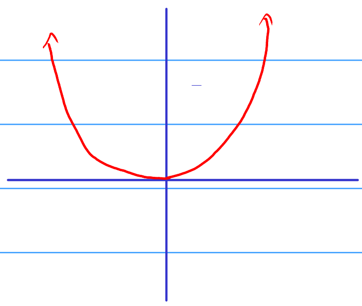
we can choose δ depending on both ε and c.

we need δ that works everywhere.

2. if you have a δ that works everywhere, then the same δ works at all points $c \in D$.

3. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ not unif. cts.



Choose $\varepsilon = 1$ for example.
Then is no $\delta > 0$ that guarantees

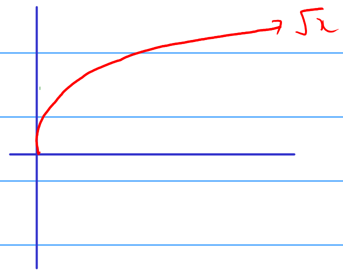
$(\forall x, y \in \mathbb{R})\ |x-y|<\delta \Rightarrow |x^2 - y^2| < 1.$

There is another version of continuity called "Lipschitz continuity":
there is some constant M such that |f(x) - f(y)| <= M |x - y| for all x, y in D.
This is stronger than uniform continuity - Lipschitz continuous implies uniformly continuous, but not necessarily the other way around.
https://en.wikipedia.org/wiki/Lipschitz_continuity

Problem 4
Decide whether each of the following functions is not continuous, continuous, or uniformly continuous. You do not have to give a formal proof.

1. $f : [0, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$.
2. $f : (0, \infty) \to \mathbb{R}, f(x) = \dfrac{1}{x}$.
3. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin(x)$.
4. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin(x) + \cos(x) + 99x$.
5. $f : \mathbb{R} \to \mathbb{R}, f(x) = x \sin(x)$.
6. $f : \mathbb{R} \to \mathbb{R}, f(x) = (x - 3\sqrt{x} + \sin(x))(\cos(x) - 727)$

1.



unif. cts.

(even though slope gets close to ∞ around 0)

Let $\varepsilon > 0$.

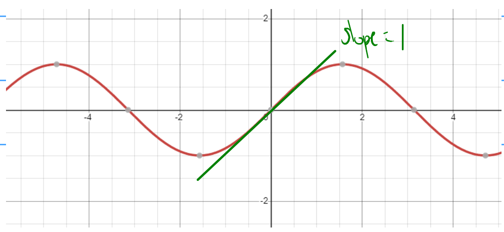
$$|\sqrt{x} - \sqrt{y}|^2 = |\sqrt{x} - \sqrt{y}|\,|\sqrt{x} - \sqrt{y}|$$
$$\overset{T.I.}{\le} |\sqrt{x} - \sqrt{y}|(|\sqrt{x}| + |\sqrt{y}|)$$
$$= |\sqrt{x} - \sqrt{y}|\,|\sqrt{x} + \sqrt{y}|$$
$$= |(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})|$$
$$= |x - y|.$$

Take $\delta = \varepsilon^2$.

Then $|x - y| < \delta \Rightarrow |\sqrt{x} - \sqrt{y}|^2 \le |x - y| < \varepsilon^2$
$\Rightarrow |\sqrt{x} - \sqrt{y}| < \varepsilon$
(both sides positive).

2. Not unif. cts,
but cts. (proof in lecture)

3. Unif cts.
$|\sin(x) - \sin(y)| < |x - y|$. (slope is bounded by 1)

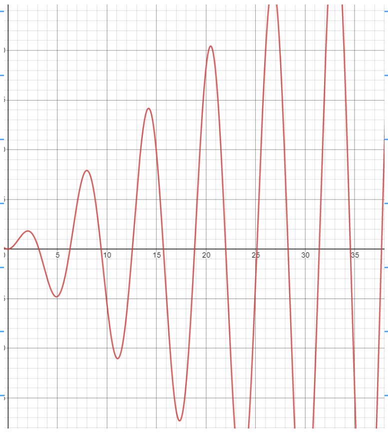


4. Unif cts.

Sum of unif cts is unif cts. ✓

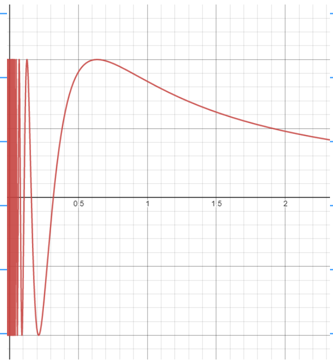
Pf is similar to non-uniform continuous case

5. Not unif cts.



✓ Same problem as x^2:

Problem 5
Give an example of a bounded and continuous function that is not uniformly continuous. You may choose the domain of this function to your liking.



$f : (0,\infty) \to \mathbb{R}$

$f(x) = \sin\left(\frac{1}{x}\right)$

exclude 0 from domain, as it's not continuous at 0

Pf that f is not unif cts.

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x,y \in (0,\infty))\left(|x-y|<\delta \text{ and } \left|\sin\left(\tfrac{1}{x}\right)-\sin\left(\tfrac{1}{y}\right)\right| \geq \varepsilon\right)$$

Let $\varepsilon = 1$. Let $\delta > 0$. Want to find $x,y \in (0,\infty)$ s.t. $|x-y|<\delta$ and $\left|\sin\left(\frac{1}{x}\right)-\sin\left(\frac{1}{y}\right)\right| \geq \varepsilon$.

idea: want x,y close enough $(|x-y|<\delta)$ with $\sin\left(\frac{1}{x}\right)=1$, $\sin\left(\frac{1}{y}\right)=0$ (so that $\left|\sin\left(\frac{1}{x}\right)-\sin\left(\frac{1}{y}\right)\right| = |1-0| = 1 \geq \varepsilon$)

Observations:

$\sin\left(\frac{1}{x}\right)=1 \iff \frac{1}{x} = \frac{\pi}{2} + 2\pi k$ for some $k \in \mathbb{N}$

$\iff x = \dfrac{1}{\frac{\pi}{2}+2\pi k}$ for some $k \in \mathbb{N}$

$\sin\left(\frac{1}{y}\right)=0 \iff \frac{1}{y} = \pi l$, $l \in \mathbb{N}$ $\iff y = \frac{1}{\pi l}$, $l \in \mathbb{N}$

Want

$$|x-y| \leq \underbrace{|x|}_{<\frac{\delta}{2}} + \underbrace{|y|}_{<\frac{\delta}{2}} < \delta$$

Choose $k \in \mathbb{N}$ large enough so that

$$\frac{1}{\frac{\pi}{2}+2\pi k} < \frac{\delta}{2}.$$

Choose $l \in \mathbb{N}$ large enough so that

$$\frac{1}{\pi l} < \frac{\delta}{2}.$$

Let $x = \dfrac{1}{\frac{\pi}{2}+2\pi k}$, $y = \dfrac{1}{\pi l}$.

We have $|x-y| \overset{\text{T.I.}}{\leq} |x| + |y| = \left|\dfrac{1}{\frac{\pi}{2}+2\pi k}\right| + \left|\dfrac{1}{\pi l}\right|$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

and $\left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{y}\right)\right| = \left|\sin\left(\frac{\pi}{2}+2\pi k\right) - \sin(\pi l)\right|$

$$= |1-0| = 1 \geq \varepsilon. \quad \text{as needed.}$$