Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = x\sin(x)$ is not uniformly continuous.

Proof. Let $\epsilon = 1$. We show that $\forall \delta > 0$, there are $x, y \in \mathbb{R}$ with $|x - y| < \delta$ and $|f(x) - f(y)| > \epsilon$.

We will use the following fact: on some interval $[0, \rho)$, we have $\sin(x) \geq \frac{1}{2}x$. The proof of this fact is not obvious, and requires additional tools (probably Taylor series). However, it should be apparent from the graph of $\sin$: in a physicist's language, $\sin(x) \approx x \geq \frac{1}{2}x$. Don't do this on your assignments.

So choose $0 < \rho < \delta$ so that $\sin(x) \geq \frac{1}{2}x$ for $x \in [0, \rho)$.

Choose $k \in \mathbb{N}$ large enough so that $2k\pi > \frac{4}{\rho}$. Choose $x = 2k\pi$, $y = 2k\pi + \frac{\rho}{2}$. We have

$$|x - y| = \frac{\rho}{2} < \rho < \delta.$$

Also,

$$y = 2k\pi + \frac{\rho}{2} > 2k\pi > \frac{4}{\rho}.$$

Thus

$$\begin{aligned}
|x\sin(x) - y\sin(y)| &= \left|x\sin(2\pi k) - y\sin\left(2\pi k + \frac{\rho}{2}\right)\right| \\
&= \left|y\sin\left(\frac{\rho}{2}\right)\right| \\
&= |y|\left|\sin\left(\frac{\rho}{2}\right)\right| \\
&\geq |y|\frac{\rho}{4} \geq \frac{4}{\rho}\frac{\rho}{4} = 1 = \epsilon.
\end{aligned}$$

This completes the proof.