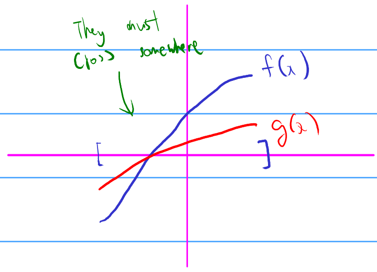
MAT157 Tutorial 7

Recall the **Intermediate Value Theorem**: if $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$, then for every $d \in [f(a), f(b)]$, there is some $c \in [a, b]$ such that $f(c) = d$. In other words, f attains all values between $f(a)$ and $f(b)$.

Problem 1
Suppose f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in $[a, b]$.



Consider $h : [a,b] \to \mathbb{R}$

$h(x) = f(x) - g(x)$.

h is continuous (as f and g are both cts)

$h(a) = f(a) - g(a) < 0$

$h(b) = f(b) - g(b) > 0$.

So $\forall\, d \in [h(a), h(b)]$, $\exists c \in [a,b]$ s.t. $h(c) = d$. (IVT)

$0 \in [h(a), h(b)]$

So $\exists\, c \in [a,b]$ s.t. $h(c) = 0$

$f(c) - g(c) = 0 \Rightarrow f(c) = g(c)$.

Problem 2

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that f only takes on integer values; in other words, $f(x) \in \mathbb{Z}$ for all $x \in [a, b]$. Using contradiction and intermediate value theorem, show that f is constant.
2. Can we conclude the same thing if instead we replaced the hypothesis $f(x) \in \mathbb{Z}$ for all $x \in [a, b]$ with $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$?

(1) 10:29

FTSOC, Suppose $f(x) \in \mathbb{Z}$ $\forall x \in [a,b]$ but f is not constant.

f not constant, so $\exists\, x_1, x_2 \in [a,b]$ s.t. $f(x_1) \neq f(x_2)$

$x_1 \neq x_2$ (since f is a function).

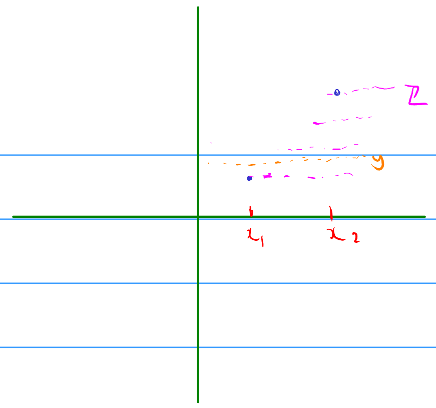
Assume $x_1 < x_2$ (otherwise we swap x_1 with x_2)

Now either ① $f(x_1) < f(x_2)$ or ② $f(x_1) > f(x_2)$

Case ①. Since $f(x) \in \mathbb{Z}$ $\forall x$,

Let $f(x_1) = k$ and $f(x_2) = l$

$k, l \in \mathbb{Z}$, $k < l$



pick a non-integer y
$k < y < l$.
(e.g. $y = k + \frac{1}{2}$)

Then since f cts on $[a,b]$
f cts on $[x_1, x_2]$
Since $k = f(x_1) < f(x_2) = l$ and $k < y < l$.
$\exists c \in [x_1, x_2]$ s.t. $f(c) = y$.
but then $f(c) \notin \mathbb{Z}$, a contradiction.

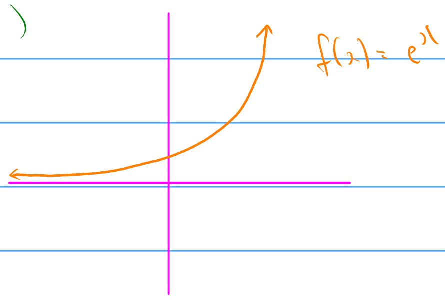
② Similar.

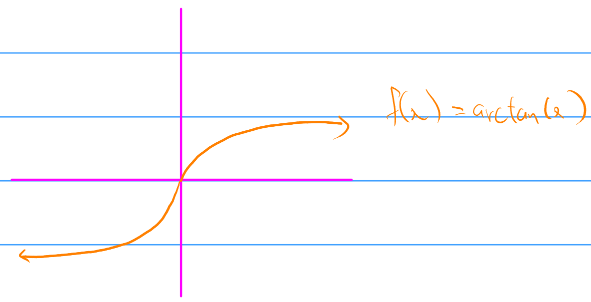
(2) Yes. Irrationals are dense, so we can repeat same proof.

Problem 3
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and increasing.

1. Show that f need not be surjective.
2. Suppose furthermore that
$$\lim_{x \to -\infty} f(x) = -\infty$$
and
$$\lim_{x \to \infty} f(x) = \infty.$$
Show that f is surjective, using the intermediate value theorem.

(1)





(2) $\lim_{x \to \infty} f(x) = \infty$ 10:52

$\Leftrightarrow (\forall M \in \mathbb{R})(\exists N \in \mathbb{R})(x > N \Rightarrow f(x) > M)$

$\lim_{x \to -\infty} f(x) = -\infty$

$\Leftrightarrow (\forall M \in \mathbb{R})(\exists N \in \mathbb{R})(x < N \Rightarrow f(x) < M)$

We show f is surjective. Let $y \in \mathbb{R}$. We want to find $x \in \mathbb{R}$ s.t. $f(x) = y$.

① $\exists N \in \mathbb{R}$ s.t. $x > N \Rightarrow f(x) > y$, since $\lim_{x \to \infty} f(x) = \infty$.
In particular, $f(N+1) > y$.

② $\exists N' \in \mathbb{R}$ s.t. $x < N' \Rightarrow f(x) < y$, since $\lim_{x \to -\infty} f(x) = -\infty$.
In particular, $f(N'-1) < y$.

Assume $N' < N$ (or else take N' smaller)

$f(N'-1) < y < f(N+1)$

f cts on $[N'-1, N+1]$

Note: f increasing not needed!

Since $y \in [f(N'-1), f(N+1)]$, by IVT, $\exists x \in [N'-1, N+1]$ s.t. $f(x) = y$.

f cts on $[N'-1, N+1]$

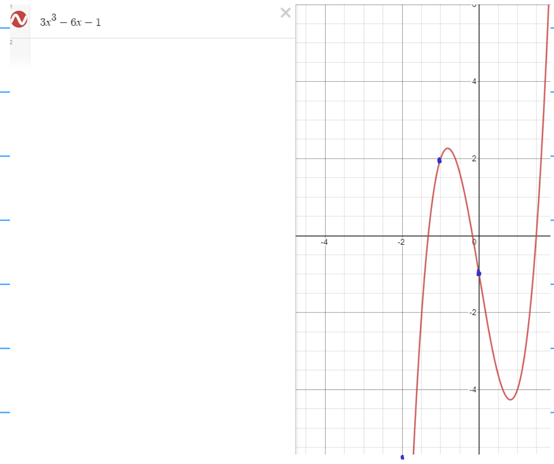
$f(N'-1) < y$, $f(N+1) > y$.

Note: increasing not needed.

By IVT, since $y \in [f(N'-1), f(N+1)]$,

$\exists x \in [N'-1, N+1]$ s.t. $f(x) = y$. $\square$

Problem 4
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x^3 - 6x - 1$. Show that the function f has at least two roots, without explicitly finding the roots.



$f(-2) = 3(-2)^3 - 6(-2) - 1$

$= -13$

$f(-1) = 3(-1)^3 - 6(-1) - 1 = 2$

$f(0) = -1$

$f(-2) < 0 < f(-1)$

f cts on $[-2, -1]$

Since $0 \in [f(-2), f(-1)]$ by IVT, $\exists\ x_1 \in [-2, -1]$ s.t. $f(x_1) = 0$.

$f(-1) > 0 > f(0)$ f cts on $[-1, 0]$

Since $0 \in [f(0), f(-1)]$, by IVT
$\exists\ x_2 \in [-1, 0]$ s.t. $f(x_2) = 0$.

$x_1 \neq x_2$: $f(x_1) = 0$ while $f(-1) \neq 0$

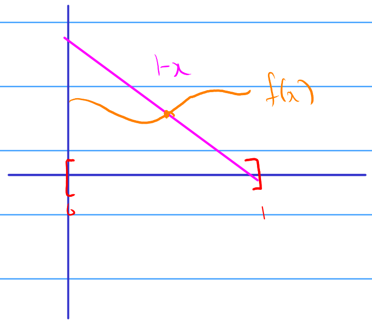
so $x_1 \in [-2, -1)$

also $x_2 \in (-1, 0]$ disjoint, hence $x_1 \neq x_2$.

Problem 5

1. Suppose that f is a continuous function on $[0,1]$ and that $f(x) \in [0,1]$ for each x. Prove that $f(x) = 1 - x$ for some number x.
2. Suppose f is as in the previous subproblem, and g is continuous on $[0,1]$ with $g(0) = 1$, $g(1) = 0$. Show $f(x) = g(x)$ for some x using a similar procedure.

(1) 11:32



define $h : [0,1] \to \mathbb{R}$

$h(x) = f(x) - (1-x)$. $\quad$ h cts on $[0,1]$

$h(0) = f(0) - 1 \leq 0$ since $f(x) \in [0,1]$ $\forall x$.

$h(1) = f(1) - (1-1) \geq 0$ since $f(x) \in [0,1]$ $\forall x$.

by IVT, since $0 \in [h(0), h(1)]$, $\exists\, x \in [0,1]$ s.t. $h(x) = 0$.

$h(x) = 0 \Rightarrow f(x) = 1 - x$.

(2) Same procedure, replace $1-x$ with $g(x$