MAT157 Tutorial 8

Given a function $f : I \to \mathbb{R}$ and $c \in I$, we say that f is **differentiable** at c if the limit

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

exists. If it exists, we define the **derivative** of f at c, written $f'(c)$, to be the above limit. We say f is **differentiable** if it is differentiable at all $c \in I$. In this case, we can define the **derivative function** (or simply **derivative**) $f' : I \to \mathbb{R}$ by $x \mapsto f'(x)$.

Recall some properties of the derivative we have shown:

- The derivative is linear: if f and g are both differentiable at c, then so is $f + g$, and $(f + g)'(c) = f'(c) + g'(c)$. If f is differentiable at c and $\alpha \in \mathbb{R}$, then so is αf, and $(\alpha f)'(c) = \alpha f'(c)$.
- Power rule: if $f : I \to \mathbb{R}$ is defined by $f(x) = x^n$ with $n \in \mathbb{N}$, then f is differentiable and
$$f'(x) = nx^{n-1}.$$
- Product rule: if f and g are both differentiable at c, then so is fg, with
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$
- Quotient rule: if f and g are both differentiable at c, and $g(c) \neq 0$, then so is f/g, with
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Problem 1
Find the derivative of the following functions.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 157x + 10^{48}$.
2. $f : \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{x^3 - 157x + 10^{48}}{2 + x^4}$.
3. $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}, f(x) = \dfrac{x^3 - \frac{157}{x-1} + 10^{48}}{2 + x^4}$.

<u>Don't Simplify</u>

1. $3x^2 - 157$.

2. $$\frac{(3x^2-157)(2+x^4) - (x^3-157x+10^{48})(4x^3)}{(2+x^4)^2}$$

3. $$\frac{d}{dx}\left(x^3 - \frac{157}{x-1} + 10^{48}\right) = 3x^2 + \frac{157}{(x-1)^2}$$

$$\frac{d}{dx}\frac{x^3 - \frac{157}{x-1} + 10^{48}}{2 + x^4} = \frac{\left(3x^2+\frac{157}{(x-1)^2}\right)(2+x^4) - \left(x^3 - \frac{157}{x-1} + 10^{48}\right)(4x^3)}{(2+x^4)^2}.$$

Problem 2
Let $f : \cancel{\mathbb{R}}\ (0,\infty) \to \mathbb{R}, f(x) = \sqrt{x}$. Show that

$$f'(x) = \frac{1}{2\sqrt{x}}$$

for all x using the limit definition of the derivative.

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$f'(x) = \lim_{y\to x} \frac{\sqrt{y}-\sqrt{x}}{y-x} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

10:27

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

continuity

Problem 3

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x^n}$ with $n \in \mathbb{N}$. Show that

$$f'(x) = -\frac{n}{x^{n+1}}$$

for all $x \neq 0$ using the limit definition of the derivative.

Hint: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$.

$f(x) = x^{-n}$

$f'(x) = -n(x^{-n-1})$ if we applied "power rule"

$= \frac{-n}{x^{n+1}}$

$$\lim_{y\to x} \frac{\frac{1}{y^n} - \frac{1}{x^n}}{y-x} = \lim_{y\to x} \frac{x^n - y^n}{(y-x)(x^n y^n)}$$

$$= \lim_{y\to x} \frac{1}{x^n y^n}\left(\frac{x^n - y^n}{y-x}\right)$$

$$= -\lim_{y\to x} \frac{1}{x^n y^n}\left(\frac{x^n-y^n}{x-y}\right)$$

$$= -\lim_{y\to x} \frac{1}{x^n y^n}\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)$$

continuity $$= -\frac{1}{x^n x^n}\left(x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1}\right)$$

$$= -\frac{1}{x^{2n}} \, n \, x^{n-1} = -\frac{n}{x^{n+1}}.$$

Problem 4 (Generalized Product Rule)

Suppose $f_1, f_2, \ldots, f_n$ are functions that are all differentiable at c. Let f be the product of all f_i:

$$f = \prod_{i=1}^{n} f_i.$$

Show that

$$f'(c) = \sum_{i=1}^{n} \left(f_i'(c) \prod_{j=1, j\neq i}^{n} f_j(c) \right).$$

~ ||:||

What does this say if $n = 2$? *Hint: Use induction and the product rule.*

$\boxed{(fg)' = f'g + fg'}$ Product rule

$$(fgh)' = (fg)'h + (fg)h' = \boxed{f'}gh + f\boxed{g'}h + fg\boxed{h'}$$

$$\frac{d}{dx}\left((x-a_1)(x-a_2)\cdots(x-a_n) \right)$$

$$= 1\cdot(x-a_2)\cdots(x-a_n) + (x-a_1)\cdot 1 \cdot \cdots \cdot (x-a_n) + \cdots (x-a_1)(x-a_2)\cdots(x-a_{n-1})\cdot 1$$

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Base case $(n=1)$: $\quad f = f_1$

$$f' = f_1' = \sum_{i=1}^{1} f_i' \prod_{j=1,\, j\neq i}^{1} f_j$$

empty product $= 1$

Thus, base case holds.

Induction step: assume $\left(\prod_{i=1}^{n} f_i\right)' = \sum_{i=1}^{n} f_i'(c) \prod_{j=1,\, j\neq i}^{n} f_j(c)$

Now let $f = \prod_{i=1}^{n+1} f_i = \left(\prod_{i=1}^{n} f_i\right) f_{n+1}$.

So $f' = \left(\prod_{i=1}^{n} f_i\right)' f_{n+1} + \left(\prod_{i=1}^{n} f_i\right) f_{n+1}'$ (product rule)

$$= \left(\sum_{i=1}^{n} f_i' \prod_{j=1,\, j\neq i}^{n} f_j\right) f_{n+1} + \left(\prod_{i=1}^{n} f_i\right) f_{n+1}' \quad \text{(IH)}$$

$$= \left(\sum_{i=1}^{n} f_{n+1} f_i' \prod_{j=1,\, j\neq i}^{n} f_j\right) + \left(\prod_{i=1}^{n} f_i\right) f_{n+1}'$$

$$= \left(\sum_{i=1}^{n} f_i' \prod_{j=1,\, j\neq i}^{n+1} f_j\right) + f_{n+1}' \prod_{j=1,\, j\neq n+1}^{n+1} f_j$$

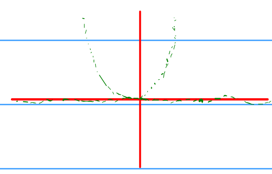
$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} a_i + a_{n+1}$ $\rightarrow$ $= \sum_{i=1}^{n+1} f_i' \prod_{j=1,\, j\neq i}^{n+1} f_j$ which shows the statement.

Problem 5
Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x^2 & x \in \mathbb{Q}. \end{cases}$$

On which points (if any) is g differentiable? Where is g non-differentiable?

Where does $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ exist?



Ans: exists only at 0.

Show ① $\lim_{h\to 0} \frac{g(h)-g(0)}{h}$ exists $\quad$ 11:27.

② g not differentiable at any $x \neq 0$. $\leftarrow$ should be a one-liner!

Pf ① $\lim_{h\to 0} \frac{g(h)-g(0)}{h}$

$= \lim_{h\to 0} \frac{g(h)}{h}$.

Observe: $0 \le g(h) \le h^2 \quad \forall h$.

Thus

$$0 \le \left|\frac{g(h)}{h}\right| \le \frac{h^2}{h}$$

$\lim_{h\to 0} 0 = 0$ $\qquad \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$.

by Squeeze, $\lim_{h\to 0} \left|\frac{g(h)}{h}\right| = 0$

$\Rightarrow \lim_{h\to 0} \frac{g(h)}{h} = 0$.

So g differentiable at h and $g'(0)=0$.

② $\boxed{g \text{ not continuous at } x \neq 0}$

differentiable $\Rightarrow$ continuous

so not cts $\Rightarrow$ not diff

if $x \neq 0$

Case ① $x \in \mathbb{Q}$

$g(x) = x^2 > 0$.

Let $\varepsilon = x^2$, for any $\delta > 0$,

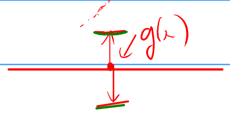
find $y \in (x-\delta, x+\delta)$, $y \notin \mathbb{Q}$.

Then $g(y) = 0$, and

$|x-y| < \delta$ and $|g(x)-g(y)| = |x^2| \ge \varepsilon$

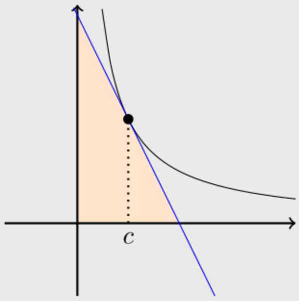
Case ② $x \notin \mathbb{Q}$

$\leftarrow g(y)$ for $y \in \mathbb{Q}$.



Problem 6

Let $c > 0$. Find the area of the triangle bounded between the x-axis, the y-axis, and the line tangent to the curve $y = \frac{1}{x}$ at c.



Eq for tangent line of f at c: 11:45

$$y = f(c) + f'(c)(x-c)$$

Eq for tangent line of $\frac{1}{x}$ at c: $(c>0)$

$$y = \frac{1}{c} + \left(\frac{1}{x}\right)'(x-c)$$

at $x = c$

$$= \frac{1}{c} - \frac{1}{c^2}(x-c)$$

y-intercept: $x = 0$

$$\frac{1}{c} - \frac{1}{c^2}(0-c) = \frac{1}{c} + \frac{1}{c} = \frac{2}{c}$$

x-intercept: $\frac{1}{c} - \frac{1}{c^2}(x-c) = 0$

$$\frac{1}{c^2}(x-c) = \frac{1}{c}$$

$$x - c = c$$

$$x = 2c.$$

Triangle area $= \frac{1}{2} bh$

$$= \frac{1}{2}(2c)\left(\frac{2}{c}\right) = 2.$$