MAT157 Tutorial 9

Problem 1

1. For $n \geq 1 \in \mathbb{N}$, consider the function

$$f_n(x) = \underbrace{\sqrt{x + \sqrt{x + \sqrt{\dots}}}}_{n \text{ times}}.$$

   For example, $f_1(x) = \sqrt{x}$ and $f_2(x) = \sqrt{x + \sqrt{x}}$. For $n \geq 1$, find a formula for $f_{n+1}(x)$ in terms of $f_n(x)$.

2. Use the chain rule to find a formula for $f'_{n+1}(x)$ in terms of $f'_n(x)$.

3. Using your previous formula, write the explicit formulas for $f'_2(x)$ and $f'_3(x)$.

1. $$f_{n+1}(x) = \underbrace{\sqrt{x + \sqrt{x + \dots}}}_{n+1 \text{ times}} = \sqrt{x + \underbrace{\sqrt{x + \dots}}_{n \text{ times}}}$$

$$= \sqrt{x + f_n(x)}$$

2. $$f_{n+1}'(x) = \left(\sqrt{x + f_n(x)}\right)'$$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

$$g(x) = x + f_n(x)$$

$$h(y) = \sqrt{y}$$

$$= \left(h(g(x))\right)'$$

$$= h'(g(x))\, g'(x)$$

$$= \frac{1}{2\sqrt{x + f_n(x)}}\left(1 + f_n'(x)\right)$$

3. $$f_1'(x) = \frac{1}{2\sqrt{x}}$$

$$f_2'(x) = \frac{1}{2\sqrt{x + f_1(x)}}\left(1 + f_1'(x)\right)$$

$$= \frac{1}{2\sqrt{x + \sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x + \sqrt{x}}} + \frac{1}{4\sqrt{x}\sqrt{x + \sqrt{x}}}$$

$$f_3'(x) = \frac{1}{2\sqrt{x + f_2(x)}}\left(1 + f_2'(x)\right)$$

$$= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}\left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} + \frac{1}{4\sqrt{x}\sqrt{x + \sqrt{x}}}\right)$$

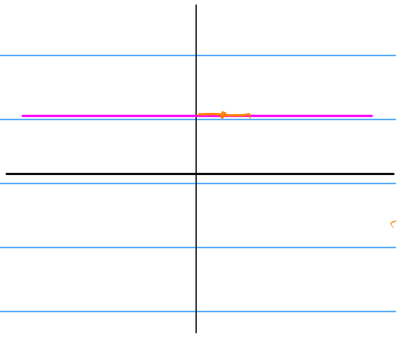
$$= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} + \frac{1}{4\sqrt{x + \sqrt{x}}\sqrt{x + \sqrt{x + \sqrt{x}}}} + \frac{1}{8\sqrt{x}\sqrt{x + \sqrt{x}}\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

The **Mean Value Theorem** states that if a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 2
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere.

1. If $f'(x) = 0$ for all $x \in \mathbb{R}$, show that f is constant.
2. If $f'(x) > 0$ for all $x \in \mathbb{R}$, show that f is strictly increasing.



1. WTS $f(x) = f(0)$ $\forall x \in \mathbb{R}$.

if $x > 0$

from MVT, $\exists\, c \in [0, x]$ $f'(c) = \frac{f(x) - f(0)}{x - 0}$

$f'(c) = 0$ by assumption

so $0 = \frac{f(x) - f(0)}{x}$

$0 = f(x) - f(0)$

so $f(x) = f(0)$

$x < 0$ similar.

Problem 3
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and suppose f' is bounded. That is, there is $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.[a]
Hint: show that if $x < y$, then $|f(x) - f(y)| \leq M|x - y|$.

[a] Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous when $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in \mathbb{R})[|x - y| < \delta \to |f(x) - f(y)| < \epsilon]$.

To show hint: ① $f(x) - f(y) \geq -M|x-y|$
② $f(x) - f(y) \leq M|x-y|$ $\Rightarrow$ $|f(x)-f(y)| \leq M|x-y|$

① $\exists\, c \in (x,y)$ s.t. $|f'(c)| \leq M \iff -M \leq f'(c) \leq M$.

$$\frac{f(y)-f(x)}{y-x} = f'(c) \leq M$$
$$f(y) - f(x) \leq M(y-x)$$
$$f(x) - f(y) \geq -M(y-x)$$
$$f(x) - f(y) \geq -M|x-y|$$

② $\exists\, c \in (x,y)$ s.t.

$$\frac{f(y)-f(x)}{y-x} = f'(c) \geq -M$$
$$f(y) - f(x) \geq -M(y-x)$$
$$f(x) - f(y) \leq M(y-x)$$
$$f(x) - f(y) \leq M|x-y|$$

To show f unif. cts.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{M}$. If $|x-y| < \delta$
$$|f(x) - f(y)| \leq M|x-y| < M\delta = \varepsilon. \checkmark$$

The **Inverse Function Theorem** states that if a function $f : \mathbb{R} \to \mathbb{R}$ is C^1 (differentiable with continuous derivative), and $f'(p) \neq 0$, then there exists open intervals U containing p and V containing $f(p)$ such that

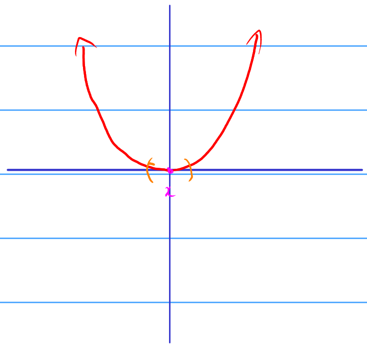
$$\hat{f} : U \to V, \hat{f}(x) = f(x)$$

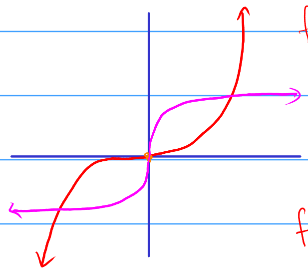
is bijective, and $\hat{f}^{-1}$ is C^1 with

$$(\hat{f}^{-1})'(y) = \frac{1}{\hat{f}'(\hat{f}^{-1}(y))}.$$

In other words, f is *locally* invertible, and its local inverse's derivative can be computed using the above formula.

Problem 4
Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(p) = 0$ yet f is still locally invertible. What is its derivative?





$f(x) = x^3$

$f'(0) = 3(0)^2 = 0$.

f bijective,

$f^{-1}(x) = \sqrt[3]{x}$

$\sqrt[3]{x} = x^{\frac{1}{3}}$

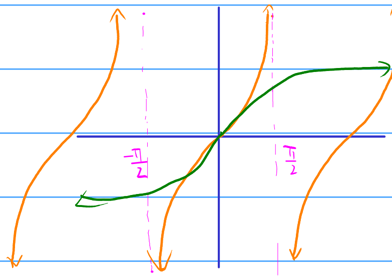
$(\sqrt[3]{x})' = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

$f^{-1\prime}(x) = \frac{1}{3x^{\frac{2}{3}}}$

does not exist at $x = 0$.

Problem 5
We define $\arctan : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ as the inverse of tan with its domain restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to ensure injectivity. Using the Inverse Function Theorem, find the derivative of arctan.

$\tan = \frac{\sin}{\cos}$

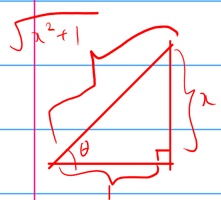


$\arctan'(x) = \frac{1}{\tan'(\arctan(x))}$

$= \frac{1}{\sec^2(\arctan(x))}$

$= \frac{1}{(\sqrt{x^2+1})^2}$

$= \frac{1}{x^2+1}$



$\sec(\arctan(x))$

$= \sec(\theta)$

$= \frac{1}{\cos(\theta)}$

$= \frac{1}{\frac{1}{\sqrt{x^2+1}}}$

$= \sqrt{x^2+1}$

$(\tan)' = \left(\frac{\sin}{\cos}\right)' = \frac{\sin' \cos - \sin \cos'}{\cos^2}$

$= \frac{\cos^2 + \sin^2}{\cos^2} = \frac{1}{\cos^2} = \sec^2$.

Problem 6

Consider the function

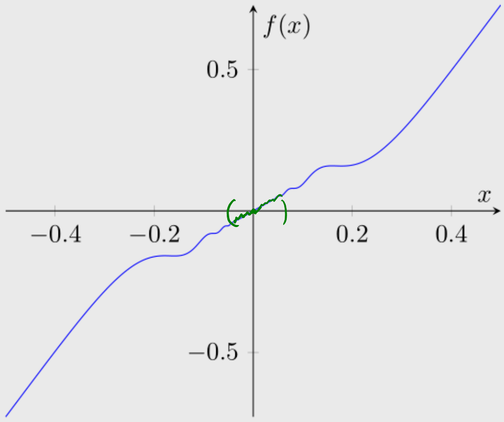
$$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

1. Find the derivative of f at 0.
2. Find the derivative of f elsewhere. This, combined with 1, shows that f is differentiable.
3. Show that 11:36

$$\lim_{x \to 0} f'(x) \neq f'(0).$$

   Conclude that f is not C^1.

It turns out that this function is not locally invertible: given any $\rho > 0$, the restriction of f to $(-\rho, \rho)$ is not injective. This is why it is necessary that we assume f is C^1 in the Inverse Function Theorem.



1.

$$f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h}$$

$(h \neq 0$ when taking limit$)$
$$= \lim_{h \to 0} \frac{h + h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h \to 0} 1 + h \sin\left(\frac{1}{h}\right) = 1$$

$$\boxed{\begin{array}{l} -h \leq h \sin\left(\frac{1}{h}\right) \leq h \\ \lim_{h \to 0} -h = 0 = \lim_{h \to 0} h \\ \text{Thus } \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0. \end{array}}$$

2. $$\frac{d}{dx}\left(x + x^2 \sin\left(\frac{1}{x}\right)\right) = 1 + 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= 1 + 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

find $\varepsilon > 0$ s.t.

$\forall \delta > 0$, $\exists x$ $|x| < \delta$ and

$\left|2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \geq \varepsilon$

3. $\lim_{x \to 0} f'(x) \neq f'(0)$

$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})\,[0 < |x| < \delta \wedge |f(x) - f(0)| \geq \varepsilon]$

Notice: $\cos(\theta) = 1 \iff \theta = 2k\pi$ for some $k \in \mathbb{Z}$.

Let $\varepsilon = \frac{1}{2}$. Let $\delta > 0$.

Choose $k \in \mathbb{N}$ large enough so that $\frac{1}{2k\pi} < \delta$.

$\cos\left(\frac{1}{x}\right) = 1 \iff \frac{1}{x} = 2k\pi$

$\iff \frac{1}{2k\pi} = x$

Then $\sin\left(\frac{1}{x}\right) = \sin(2k\pi) = 0$

$\cos\left(\frac{1}{x}\right) = \cos(2k\pi) = 1$

$\left|\underbrace{2x \sin\left(\tfrac{1}{x}\right)}_{0} - \cos\left(\tfrac{1}{x}\right)\right| = |-1| = 1 \geq \varepsilon.$

$|f'(x) - f'(0)| \geq \varepsilon.$

f' not continuous!